MATH131A - Discussion Supplements for Summer 22

Contents are motivated from [1]. [1]

Contents

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1 Discussion 1

Set of Natural Numbers $\mathbb{N}$ and Induction.

The key points about the set of natural numbers $\mathbb{N}$ from the book and lectures are:

- $\mathbb{N} = \{1, 2, 3, ...\}$
- $\mathbb{N}$ starts from 1, which might contradict certain conventions in other math textbooks (where $\mathbb{N}$ starts from 0).
- (N5) of Peano Axiom: A subset $S \subseteq \mathbb{N}$ in which[2]
  (i) $1 \in S$, and
  (ii) $n \in S$ implies $n + 1 \in S$,

  then $S = \mathbb{N}$.

The last bullet point thus allows for induction on $\mathbb{N}$, in which the rigorous idea is as follows. (You can skip this if you think that you would just want to know how to "solve" an induction problem, but I would say that understanding the idea below helps to build experiences in mathematical proofs!)

Let $P(n)$ be a proposition P depending on a natural number $n \in \mathbb{N}$ that you would like to prove to be true for all $n \in \mathbb{N}$. Now, we denote the set S to be all the natural numbers[3] such that $P(n)$ holds, ie

$$S = \{n \in \mathbb{N} : P(n) \text{ holds}\}. \tag{1}$$

If we can show that

(i) $1 \in S$, that is, $P(1)$ holds, and

(ii) $n \in S$ implies that $n + 1 \in S$, that is, $P(n)$ is true implies that $P(n + 1)$ is true,

by (N5) of Peano Axiom, we deduce that $S = \mathbb{N}$. Expanding out the definition of S in (1), this means that

$$\mathbb{N} = \{n \in \mathbb{N} : P(n) \text{ holds}\}, \tag{2}$$

in other words,

$$\mathbb{N} \subseteq \{n \in \mathbb{N} : P(n) \text{ holds}\}. \tag{3}$$

Now,

1. By the definition of $\subseteq$,[4], let $n \in \mathbb{N}$ be any given natural number.
2. By (3), this implies that $n \in \{n \in \mathbb{N} : P(n) \text{ holds}\}$, that is, $P(n)$ holds.
3. Since $n \in \mathbb{N}$ is arbitrary, then we have that $P(n)$ is true for all $n \in \mathbb{N}$.

The above is summarized in a Proposition below.

Proposition 1. Suppose we would want to prove that $P(n)$ is true for all $n \in \mathbb{N}$.
If we can show that

(i) $P(1)$ is true, and

(ii) For any given $n \in \mathbb{N}$, $P(n)$ is true implies that $P(n + 1)$ is true,

then $P(n)$ is true for all $n \in \mathbb{N}$. This is known as **proof by induction**.

The next page that follows are three examples of induction.

[2]Some books uses $\subset$ to represent "subset", while other uses $\subseteq$. Since the textbook uses the latter, we shall do so too.
[3]Naturally, the set of all natural number such that something holds is a subset of natural numbers.
[4]Recall this from Math 61 that if $A \subseteq B$, then for all $x \in A$, then $x \in B$.

Example 2. (Exercise 1.5.) Prove by induction for all positive integers n[a] that

$$\sum_{k=0}^{n} \frac{1}{2^k} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}. \tag{4}$$

[a] To be rigorous, we have not defined integers properly here. Nonetheless, we take the set of integers to be $\mathbb{Z} = \{0, 1, -1, 2, -1, \ldots\}$, and hence positive integers are just the natural numbers!

Proof.

1. Let $P(n)$ be the proposition "$\sum_{k=0}^{n} \frac{1}{2^k} = 2 - \frac{1}{2^n}$".
2. Verify that $P(1)$ is true, that is, $\sum_{k=0}^{1} \frac{1}{2^k} = 2 - \frac{1}{2^1}$. Indeed, we have

$$\begin{aligned} \text{Left Hand Side of } P(1) &= \sum_{k=0}^{1} \frac{1}{2^k} \\ &= \frac{1}{2^0} + \frac{1}{2^1} \\ &= 1 + \frac{1}{2} = \frac{3}{2} \\ &= 2 - \frac{1}{2^1} \\ &= \text{Right Hand Side of } P(1). \end{aligned} \tag{5}$$

3. Now, we want to show that for a given natural number n, if $P(n)$ is true, then $P(n+1)$ is true.
   (i) Assuming that $P(n)$ is true, ie $\sum_{k=0}^{n} \frac{1}{2^k} = 2 - \frac{1}{2^n}$.
   (ii) WTS (Want To Show) that $P(n+1)$ is true, ie $\sum_{k=0}^{n+1} \frac{1}{2^k} = 2 - \frac{1}{2^{n+1}}$.
   (iii) Indeed, we have

$$\begin{aligned} \text{Left Hand Side of } P(n+1) &= \sum_{k=0}^{n+1} \frac{1}{2^k} \\ &= \sum_{k=0}^{n} \frac{1}{2^k} + \frac{1}{2^{n+1}} && \left(\text{by definition of } \sum\right) \\ &= 2 - \frac{1}{2^n} + \frac{1}{2^{n+1}} && \text{(by induction hypothesis, 3(i))} \\ &= 2 - 2\left(\frac{1}{2^{n+1}}\right) + \frac{1}{2^{n+1}} \\ &= 2 - \frac{1}{2^{n+1}} && \text{(by simplifying the last two terms)} \\ &= \text{Right Hand Side of } P(n+1), \end{aligned} \tag{6}$$

   and thus $P(n)$ is true implies that $P(n+1)$ is true.
4. By induction, we have that $P(n)$ is true for all $n \in \mathbb{N}$, and we are done.

□

Example 3. Let $x \in \mathbb{R}$ such that $x \geq -1$ be fixed. Prove that for all positive integers, we have

$$1 + nx \leq (1+x)^n. \tag{7}$$

For this question, you do not have to rigorously justify the arithmetic steps using A1 - A4, M1 - M4 and DL for $\mathbb{R}$.

Proof.

1. Let $x \in \mathbb{R}$ such that $x \geq -1$ be fixed. Let $P(n)$ be the proposition "$1 + nx \leq (1+x)^n$".
2. Verify that $P(1)$ is true, that is, $1 + 1 \cdot x \leq (1+x)^1$. Indeed, we have

$$\begin{aligned} \text{LHS of } P(1) &= 1 + x \\ &= (1+x)^1 \\ &\leq (1+x)^1 \\ &= \text{RHS of } P(1). \end{aligned} \tag{8}$$

   in which the second equality holds since if equality holds, then $\leq$ meaning "$=$" or "$<$" holds.
3. Now, we want to show that for a given natural number n, if $P(n)$ is true, then $P(n+1)$ is true.
   (i) Assuming that $P(n)$ is true, ie $1 + nx \leq (1+x)^n$.
   (ii) WTS (Want To Show) that $P(n+1)$ is true, ie $1 + (n+1)x \leq (1+x)^{n+1}$.
   (iii) Indeed, we have

$$\begin{aligned} \text{RHS of } P(n+1) &= (1+x)^{n+1} \\ &= (1+x)(1+x)^n \\ &\geq (1+x)(1+nx) && \text{(by induction hypothesis, 3(i) and that } 1+x \geq 0) \\ &= 1 + nx + x + nx^2 && \text{by distributive law for } \mathbb{R} \\ &\geq 1 + nx + x && \text{since } n \geq 1 \text{ as natural number and } x^2 \geq 0 \\ &= 1 + (n+1)x \\ &= \text{LHS of } P(n+1). \end{aligned} \tag{9}$$

   Thus, we have that $P(n)$ is true implies that $P(n+1)$ is true.
4. By induction, we have that $P(n)$ is true for all $n \in \mathbb{N}$, and we are done.

□

Remark: You will realize that it is slightly harder to start from LHS of $P(n+1)$ but that will still work out if you can navigate through the correct logic!

Example 4. Prove the following statement by induction:

$$\forall n \in \mathbb{Z}_{\geq 8}, \exists x, y \in \mathbb{Z}_{\geq 0}, n = 3x + 5y. \tag{10}$$

Here, $\mathbb{Z}_{\geq k}$ refers to integers greater than or equals to k.

Intuitively, this statement can be translated to
"Any integer-valued transactions greater than or equals to 8 dollars can be carried out with only 3-dollar and 5-dollar notes."

Proof.

1. Using a similar idea as Proposition 1, we can show that any subset S of $\mathbb{N}_{\geq 8} = \mathbb{Z}_{\geq 8}$ with the properties
   (i) $8 \in S$ and
   (ii) For all $n \in \mathbb{N}_{\geq 8}$, $P(n)$ is true implies that $P(n+1)$ is true,

   then $S = \mathbb{N}_{\geq 8}$. (Think of the domino effect as mentioned above.) In other words, we repeat a similar argument for induction, with our starting case to be at $n = 8$ instead.

2. Let $P(n)$ be the proposition "$\exists x, y \in \mathbb{Z}_{\geq 0}, n = 3x + 5y.$".

3. Verify that $P(8)$ is true, that is, $\exists x, y \in \mathbb{Z}_{\geq 0}, 8 = 3x + 5y.$. Indeed, if we pick $x = 1 \in \mathbb{Z}_{\geq 0}$ and $y = 1 \in \mathbb{Z}_{\geq 0}$, then $8 = 3 \cdot 1 + 5 \cdot 1$ and we are done.

4. Now, we want to show that for a given natural number $n \geq 8$, if $P(n)$ is true, then $P(n+1)$ is true.
   (i) Assuming that $P(n)$ is true for $n \geq 8$, ie $\exists x, y \in \mathbb{Z}_{\geq 0}, n = 3x + 5y$.
   (ii) WTS (Want To Show) that $P(n+1)$ is true, ie $\exists x', y' \in \mathbb{Z}_{\geq 0}, n + 1 = 3x' + 5y'$.
   (iii) Intuitively, for a given configuration making up n dollars made of up $3-$ and $5-$ dollar notes, one can either trade $3\times$ $3-$dollar notes for $2\times$ $5-$dollar note to add a dollar. However, this assumes that we have at least $3\times$ $3-$dollar notes in the first place. Else, this implies that we have at most $2 \times 3-$dollar notes. For $n \geq 8$ dollars, we must then have at least $1 \times 5-$dollar note. For this case, we can instead trade in the $1 \times 5-$dollar note for $2 \times 3-$dollar note to gain a dollar and conclude. [5]
   (iv) For a rigorous proof, we split it up into two cases.
      a. If $x \geq 3$, then we see that
      $$\begin{aligned} n + 1 &= 3x + 5y + 1 \\ &= 3(x-3) + 5(y+2) \\ &= 3x' + 5y'. \end{aligned} \tag{11}$$
      Thus, there exists $x', y' \in \mathbb{Z}_{\geq 0}$ such that $n + 1 = 3x' + 5y'$, These choices are valid since the given $y \in \mathbb{Z}_{\geq 0}$, and so is $y' = y + 2 \in \mathbb{Z}_{\geq 0}$; and since $x \geq 3$, we have $x - 3 \geq 0$ and thus $x' = x - 3 \in \mathbb{Z}_{\geq 0}$.
      b. Else, $x \leq 2$. Since $n = 3x + 5y$ and $x \leq 2$ while $n \geq 8$, we must then have $y \geq 1$. Then, observe that
      $$\begin{aligned} n + 1 &= 3x + 5y + 1 \\ &= 3(x+2) + 5(y-1) \\ &= 3x' + 5y'. \end{aligned} \tag{12}$$
      where $x' = x + 2$ and $y' = y - 1 \geq 0$ are valid choices since $y \geq 1$.

   Thus, we have that $P(n)$ is true implies that $P(n+1)$ is true.

5. By induction, we have that $P(n)$ is true for all $n \in \mathbb{N}_{\geq 8} = \mathbb{Z}_{\geq 8}$, and we are done.

$\square$

[5] In lectures, the professor mentioned that giving a rough idea of what is going on can yield some partial credits given that your proof might be mathematically incorrect. Thus, it would help to write down the main intuitive idea of your proof to show that you know how to approach any given problem in this class!

Set of Real Numbers $\mathbb{R}$, Fields, and Ordered Fields.

Definition 5. A **field** $\mathbb{F}$ equipped with two operations $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$, and in the presence of its additive and multiplicative identities 0 and 1 (where $0 \neq 1$), is an algebraic system satisfying the following axioms for all $a, b, c \in \mathbb{F}$:

A1. $a + (b + c) = (a + b) + c$.

A2. $a + b = b + a$.

A3. $a + 0 = a$.

A4. There exists $-a \in \mathbb{F}$ such that $a + (-a) = 0$.

M1. $a(bc) = (ab)c$.

M2. $ab = ba$.

M3. $a \cdot 1 = a$.

M4. There exists $a^{-1} \in \mathbb{F}$ such that $aa^{-1} = 1$.

DL. $a(b + c) = ab + ac$. (Distributive Law)

Definition 6. An **ordered field** $\mathbb{F}$ is a field (that is, with axioms listed in Definition 5) and with the following order structure $\leq$ for any $a, b \in \mathbb{F}$:

O1. Either $a \leq b$ or $b \leq a$.

O2. If $a \leq b$ and $b \leq a$, then $a = b$.

O3. If $a \leq b$ and $b \leq c$, then $a \leq c$. (Transitive Property)

O4. If $a \leq b$, then $a + c \leq b + c$.

O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Some important examples includes:

- $\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$ is an ordered field.
- $\mathbb{R}$, the set of real numbers, is an ordered field.
- $\mathbb{C}$, the set of complex numbers, if a field but is not an ordered field (since we can't decide if $\mathrm{i} \leq 2$ or $2 \leq \mathrm{i}$).

Some properties of elements in a field are summarized below.

Theorem 7. For any $a, b, c \in \mathbb{F}$ (not necessarily ordered),

(i) $a + c = b + c$ implies $a = b$.

(ii) $a \cdot 0 = 0$.

(iii) $(-a) \cdot b = -(a \cdot b)$.

(iv) $(-a)(-b) = ab$.

(v) $ac = bc$ and $c \neq 0$ implies $a = b$.

(vi) $ab = 0$ implies either $a = 0$ or $b = 0$.

Some properties of elements in a **ordered** field are summarized below.

Theorem 8. For any $a, b, c \in \mathbb{F}$ (must be ordered),

(i) If $a \leq b$, then $-b \leq -a$.

(ii) If $a \leq b$ and $c \leq 0$, then $bc \leq ac$.

(iii) If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.

(iv) $0 \leq a^2$.

(v) $0 < 1$.

(vi) If $0 < a$, then $0 < a^{-1}$.

(vii) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

Here, $a < b$ means $a \leq b$ **and** $a \neq b$.

For more information on the relevant proofs of these theorems using the relevant axioms in the definitions above, refer to the textbook. We shall now restrict our attention to $\mathbb{F} = \mathbb{R}$, the set of real numbers as an ordered field.

Definition 9. Given $a \in \mathbb{R}$, we define

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0 \end{cases} \tag{13}$$

as the **absolute value** of a. Furthermore, we define

$$\text{dist}(a, b) = |a - b| \tag{14}$$

where dist(a, b) represents the **distance** between a and b.

The following theorem records the relevant properties of absolute values and distances.

Theorem 10. For all $a, b, c \in \mathbb{R}$, we have

(i) $|a| \geq 0$.

(ii) $|ab| = |a| \cdot |b|$.

(iii) $|a + b| \leq |a| + |b|$. (Triangle Inequality)

(iv) $\text{dist}(a, c) \leq \text{dist}(a, b) + \text{dist}(b, c)$. (Triangle Inequality too)

The following are some examples on proving relevant properties of $\mathbb{R}$ as an ordered field.

Example 11. (Exercise 3.5(a).) For any $a, b \in \mathbb{R}$, prove that $|b| \leq a$ if and only if $-a \leq b \leq a$.

Proof. Here, "if and only if" here means that you have to prove both directions.

1. Before we prove anything, we prove the following lemma: $-|x| \leq x \leq |x|$ for any $x \in \mathbb{R}$.
   (i) By reserving the role of $|b|$ and b, by (13) in Definition 9, we see that
   $$b = \begin{cases} |b| & \text{if } b \geq 0 \\ -|b| & \text{if } b \leq 0. \end{cases} \tag{15}$$
   (ii) Thus, b is either $|b|$ or $-|b|$.
   (iii) By Theorem 10 (i), $|b| \geq 0$.
   (iv) By Theorem 8 (i), this implies that $-|b| \leq -0 = 0$, where the last equality follows from Theorem 7(ii).
   (v) By Definition 6 O3. (transitivity of $\leq$), we deduce that $-|b| \leq 0 \leq |b|$.
   (vi) Using (ii), if $b \geq 0$, then $b = |b| \leq |b|$ and $b = |b| \geq 0 \geq -|b|$.
   (vii) Else, if $b \leq 0$, then $b = -|b| \geq -|b|$ and $b = -|b| \leq 0 \leq |b|$.
   (viii) We then deduce that $-|b| \leq b \leq |b|$ since this is true in both cases.
2. We also prove the following lemma: $-(-x) = x$ for any $x \in \mathbb{R}$.
   (i) We start from Theorem 7(iv), with $a = 1$ and $b = x$. This implies that we have $(-1)(-x) = 1 \cdot x = x$ (by M3. of Definition 5).
   (ii) Now, apply Theorem 7(iii), with $b = -x$ and $a = 1$. This implies that $(-1)\cdot(-x) = -(1\cdot(-x)) = -(-x)$. (by M3. of Definition 5, since $-x \in \mathbb{R}$). This concludes the proof!
3. Suppose that $|b| \leq a$.
   (i) Apply 1. with $x = b$ to obtain $-|b| \leq b \leq |b|$.
   (ii) Since $|b| \leq a$, by transitivity of $\leq$ (Definition 6 O3.), we have $-|b| \leq b \leq |b| \leq a$, and thus $b \leq a$ (here, transitivity is applied again).
   (iii) On the other hand, since $0 \leq |b| \leq a$ (by Theorem 10 (i)), we deduce that $a \geq 0$ by transitivity of $\leq$.
   (iv) By Theorem 8 (i) and Theorem 7 (ii) (similar to 1.(iv) above), we deduce that $-a \leq 0$.

   Now, split into two cases.
   (v) <u>If $b \geq 0$</u>, then $|b| = b$.
   (vi) Use (ii) and transitivity of $\leq$ to deduce that $-b \leq a$.
   (vii) Use Theorem 8 (i) to deduce that $b = -(-b) \geq -a$. Here, we have used the fact that $-(-b) = b$ from 2. (by setting $x = b$).
   (viii) <u>If $b \leq 0$</u>, then $|b| = -b$.
   (ix) Use (ii) and transitivity of $\leq$ to deduce that $-b = |b| \leq a$.
   (x) A similar argument as in (vii) can be used to deduce that we also have $b \geq -a$ for this case.
   (xi) All in all, we have $-a \leq b \leq a$.
4. We are not done yet! Since this is an "if and only if" statement, we have to prove the other direction. Suppose that $-a \leq b \leq a$.
   (i) If <u>$b \geq 0$</u>, then $b = |b|$.
   (ii) Substitute this into the assumption, we have $-a \leq |b| \leq a$. In particular, we have $|b| \leq a$.
   (iii) If <u>$b \leq 0$</u>, then $b = -|b|$.
   (iv) This implies that $-a \leq -|b| \leq a$. In particular, we have $-a \leq -|b|$.
   (v) By Theorem 8 (i) and the lemma in 2., we thus have $|b| \leq a$.
   (vi) In both cases, we thus have $|b| \leq a$ and we are done.

Example 12. (Exercise 3.6.)

(a) Prove that $|a+b+c| \leq |a|+|b|+|c|$ for all $a, b, c \in \mathbb{R}$.

(b) Use induction to prove that

$$|a_1+a_2+\cdots+a_n| = \left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k| = |a_1|+|a_2|+\cdots+|a_n|. \tag{16}$$

for any n real numbers $a_1, a_2, \cdots, a_n$, where n is a natural number.

Proof:

(a) This follows by applying Triangle Inequality (Theorem 10 (iii), $|x+y| \leq |x|+|y|$) twice as follows:

$$\begin{aligned} |a+(b+c)| &\leq |a|+|b+c| && \text{First Application; } x=a,\ y=b+c \\ &\leq |a|+(|b|+|c|) && \text{Second Application; } x=b,\ y=c \\ &= |a|+|b|+|c|. \end{aligned} \tag{17}$$

(b) Let $P(n)$ be the proposition "$|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k|$ for any real numbers $a_1, a_2, \cdots, a_n$".

1. Verify that $P(1)$ is true, that is, $\left|\sum_{k=1}^{1} a_k\right| = \sum_{k=1}^{1} |a_k|$, which follows from the fact that $\left|\sum_{k=1}^{1} a_k\right| = |a_1| = \sum_{k=1}^{1} |a_k|$.
2. Now, we want to show that for a given natural number n, if $P(n)$ is true, then $P(n+1)$ is true.
   (i) Assuming that $P(n)$ is true, ie, $|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k|$ for any real numbers $a_1, a_2, \cdots, a_n$.
   (ii) WTS (Want To Show) that $P(n+1)$ is true, ie, $\left|\sum_{k=1}^{n+1} a_k\right| \leq \sum_{k=1}^{n+1} |a_k|$ for any real numbers $a_1, a_2, \cdots, a_{n+1}$.
   (iii) Indeed, we have

$$\begin{aligned} \text{RHS of } P(n+1) &= \left|\sum_{k=1}^{n+1} a_k\right| \\ &= \left|\sum_{k=1}^{n} a_k + a_{n+1}\right| \\ &\leq \left|\sum_{k=1}^{n} a_k\right| + |a_{n+1}| && \text{by Triangle inequality; } x=\sum_{k=1}^{n} a_k,\ y=a_{n+1} \\ &\leq \sum_{k=1}^{n} |a_k| + |a_{n+1}| && \text{by induction hypothesis, 2(i)} \\ &= \sum_{k=1}^{n+1} |a_k| && \text{by definition of } \sum_{k=1}^{n+1} \\ &= \text{LHS of } P(n+1). \end{aligned} \tag{18}$$

   Thus, we have that $P(n)$ is true implies that $P(n+1)$ is true.
3. By induction, we have that $P(n)$ is true for all $n \in \mathbb{N}$, and we are done.

Proof Techniques for Quantified Statements.

Quantifiers.

In math, we use $\exists$ to represent the English phrase "there exists", and $\forall$ to represent the English phrase "for all". For instance, if we want to claim that there is a solution to the algebraic equation

$$x + 1 = 2,$$

we say in symbols that

$$\exists x, x + 1 = 2,$$

which reads "there exists an x such that $x + 1 = 2$." Furthermore, one should note that we tend to use a comma (,) to mean "such that".

We then introduce the ability to **negate** statements. We denote the negation of a statement as $\neg$. Thus, if we want to say that the statement

$$\exists x,\ x + 1 = 2$$

is false, this is equivalent to saying that its negation is true. To conduct the negation, we use what is known as the De Morgan's Law, in which $\neg(\exists) = \forall$, $\neg(\forall) = \exists$, and mathematical claims are negated accordingly. Indeed, we see that

$$\neg(\exists x,\ x + 1 = 2) = \forall x, x + 1 \neq 2,$$

which if you think about it, makes sense! If there does not exists an x that solves $x + 1 = 2$, then it means that for every possible x, $x + 1 \neq 2$.

Note that what each of these symbols mean individually is usually intuitively clear, but what they mean when they are combined tends to throw beginning students in proof-based courses off. The key here is in the **order in which they are presented**. To conceptualize this, we consider the following example:

Example 13. Which of the following statements is/are true?

(i) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 1.$

(ii) $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x + y = 1.$

(iii) $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 1.$

(iv) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x + y = 1.$

Suggested Solution:

(i) If $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 1$ is true, this means that "for all $x \in \mathbb{Z}$, there exists a $y \in \mathbb{Z}$ such that $x + y = 1$". In other words, this implies that for any given $x \in \mathbb{Z}$, we can always find a $y \in \mathbb{Z}$ such that $x + y = 1$. At the "always find a $y \in \mathbb{Z}$" part, we are assuming that the value of x is already given, though if you want it to hold "for all" x, your argument must work for any possible x.

True. Indeed, if $x \in \mathbb{Z}$ is given to me, I can pick $y = -x + 1$. Indeed[6], we have $x + y = x + (-x) + 1 = 1$.

(ii) For $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x + y = 1$ to be true, this means that "there exists $x \in \mathbb{Z}$, such that for all $y \in \mathbb{Z}$, $x + y$ will be equals to 1." Based on the given sequence, at the point in which the question on "existence of x" is asked, no information about $y \in \mathbb{Z}$ is known. In fact, you must pick an x such that no matter what values of $y \in \mathbb{Z}$ were thrown at you, you will always have $x + y = 1$. With this new angle of looking at the problem, it is now intuitively true that the answer to this is false!

False. Note that it is hard to prove that something is False, but it would be easier to prove that its negation is True (which means that the original statement is False). Negating the statement, we obtain[7]

$$\neg(\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x + y = 1) \equiv \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y \neq 1.$$

[6] You can be more rigorous by citing the relevant axioms/theorems on why $x + (-x) = 0$, but this is clearly not the objective of this question! In fact, for most cases, you would not have to cite these axioms, unless a question looks like it is actually asking you to deduce a certain property of say $\mathbb{Z}$ axiomatically.

[7] We use $\equiv$ for logical equivalence.

Using the style that we have proved (i) to be True, we just have to prove that the above negation is true. Given any $x \in \mathbb{Z}$, we now pick $y = -x$ (recall that since the existence question is asked second, after information on x is revealed, we can always pick y depending on x). We can then check that $x + y = x + (-x) = 0 \neq 1$, and hence the negation is True (and thus the original statement is False)!

(iii) $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 1.$
True. Indeed, pick $x = 0 \in \mathbb{Z}$ and $y = 1 \in \mathbb{Z}$, and one can see that $0 + 1 = 1$.

Note that if existential questions are asked one after the other, though the sequence still matters, we are in control of both "exists", so information on x is available and thus can be picked when asked for existence of y. In other words, a valid proof would be pick $x \in \mathbb{Z}$ to be any integer and let $y = -x + 1$.

Does this sounds familiar? This is exactly the same proof as (i). Indeed, if we view (i) as "for all $x \in \mathbb{Z}$ such that some statement is true", and (iii) as "there exists $x \in \mathbb{Z}$ such that the same statement is true", then (i) implies (iii), since we can submit any x generated from (i) to (iii).[8]

(iv) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x + y = 1.$

False. Similar to (ii), it might be easier to prove the negation of the statement to be True. Indeed, we have

$$\neg(\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x + y = 1) \equiv \exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y \neq 1.$$

Indeed, pick $x = 0 \in \mathbb{Z}$ and $y = 0 \in \mathbb{Z}$, we can then see that $0 + 0 = 0 \neq 1$, and thus the negation of the statement is True!

Here are some further remarks:

- As a rule of thumb, the more $\exists$ you see, the more it asks you to be creative and it is likely that the statement is true (as to satisfy $\exists$, only an example is required).
- The more $\forall$ you see, the more likely that the statement is false, as you have to "defend" against all possible cases thrown at you!

[8]Caution: This depends that $\mathbb{Z}$ must have at least some element, but we shall not talk about such cases as of now since intuitively, $\mathbb{Z}$ is non-empty.

2 Discussion 2

Rational Numbers $\mathbb{Q}$.

Here are some key results that you should know for rational numbers:

- By definition, the set of rational numbers $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$.
- (Theorem 2.2, Rational Zeros Theorem.) For integers $c_0, c_1, \cdots c_n$ (with $c_n \neq 0$) and rational number r satisfying the polynomial equation

$$c_n x^n + c_{n-1} x^{n-1} + \cdots c_1 x + c_0 = 0, \tag{19}$$

  then r must be of the form:

$$r = \pm \frac{\text{factors of } c_0}{\text{factors of } c_n}. \tag{20}$$

- (Corollary 2.3.) In particular, if $c_n = 1$, then r must be of the form:

$$r = \pm \text{ factors of } c_0. \tag{21}$$

One can show that the above version of Theorem 2.2 and Corollary 2.3 is equivalent to that in the textbook.

Example 14. (Exercise 2.3.) Show that $\sqrt{2+\sqrt{2}}$ is not a rational number.

Proof.

1. Let $x = \sqrt{2+\sqrt{2}}$ and suppose for a contradiction that x is rational.
2. Next, we shall determine the polynomial equation satisfied by x. Observe that

$$\begin{aligned} x &= \sqrt{2+\sqrt{2}} \\ x^2 &= 2+\sqrt{2} \\ x^2 - 2 &= \sqrt{2} \\ (x^2-2)^2 &= 2 \\ x^4 - 4x^2 + 2 &= 0. \end{aligned} \tag{22}$$

3. By Corollary 2.3, we see that since x is a rational number (and the coefficients $1, 0, -4$, and 2 are integers), we obtain that $x = \pm 1, \pm 2$ since 2 only has two factors, 1 and 2.
4. We then check that $x = \pm 1, \pm 2$ does not satisfy (22), as follows
   - At $x = \pm 1$, we have $(\pm 1)^4 - 4(\pm 1)^2 + 2 = -1 \neq 0$.
   - At $x = \pm 2$, we have $(\pm 2)^4 - 4(\pm 2)^2 + 2 = 2 \neq 0$.
5. 4. contradicts Corollary 2.3 in 3., and thus we have a contradiction. This implies that our initial assumption must be false. Thus, $x = \sqrt{2+\sqrt{2}}$ is irrational.

□

The proof might seem a little arcane and artifical, but one can try to see when the proof fails for $x = \sqrt{4}$. Indeed, for that, the polynomial equation satisfied by x is $x^2 - 4 = 0$. By the Corollary, we see that x then must be equals to either $\pm 1, \pm 2$, or ± 4. We can then check that $x = \pm 2$ are indeed solutions to $x^2 - 4 = 0$, so we cannot obtain a contradication here.

Supremums, Infimums, $\pm\infty$, and Completeness of $\mathbb{R}$.

Here are some key definitions. Let S be a non-empty subset of $\mathbb{R}$.

- The **maximum** of set S refers to an element $s_0 \in S$ such that $s \leq s_0$ for all $s \in S$. We denote this maximal element $s_0 := \max S$.
- The **minimum** of set S refers to an element $s_0 \in S$ such that $s \geq s_0$ for all $s \in S$. We denote this minimal element $s_0 := \min S$.
- An **upper bound** of set S refers to a real number M such that $s \leq M$ for all $s \in S$. (Remark: M does not have to be in S.) If a set S has an upper bound, we say that S is **bounded (from) above**.
- A **lower bound** of set S refers to a real number m such that $s \geq m$ for all $s \in S$. (Remark: m does not have to be in S.) If a set S has a lower bound, we say that S is **bounded (from) below**.
- We then say that the set S is **bounded** if it is bounded above and bounded below. Mathematically, we say that a set S is bounded if there exists real numbers m and M such that $S \subseteq [m, M]$.
- If S is bounded from above, then the least upper bound exists, and we denote the least upper bound or **supremum** as $\sup S$. If S is not bounded from above, we denote $\sup S = +\infty$.
- If S is bounded from below, then the greatest lower bound exists, and we denote the greatest lower bound or **infimum** as $\inf S$. If S is not bounded from below, we denote $\inf S = -\infty$.

Here are some remarks with regards to the definition.

- In the textbook, it is mentioned that if a non-empty subset of $\mathbb{R}$, S, is bounded from above and that there exists a least upper bound, then we denote it as $\sup S$. However, the existence of the least upper bound is guaranteed by the Completeness Axiom (Theorem 4.4.), and thus we have removed it from the definition.
- Theorem 4.4 as mentioned above is known as the "completeness" of $\mathbb{R}$. This might not hold in $\mathbb{Q}$ since for a non-empty subset of $\mathbb{Q}$, the least upper bound might not be in $\mathbb{Q}$ (see $\{x \in \mathbb{Q} : x < \sqrt{2}\}$).
- Note that we have introduced the notion of $\pm\infty$ above, but we have not talked about what exactly it is. One way to think about this is that $+\infty$ and $-\infty$ are not elements of $\mathbb{R}$, but we can always include $\pm\infty$ to consider the **extended real line** (denoted by $\mathbb{R} \cup \{-\infty, +\infty\}$). Though not mentioned in the book, we can think of what the symbols meant intuitively as being rigorous in the following sense: $\pm\infty$ satisfies the following properties:
  - $+\infty$ is usually denoted as ∞ for convenience.
  - For any $a \in \mathbb{R}$, $a + (\infty) = \infty$.
  - For any $a \in \mathbb{R}$, $a + (-\infty) = a - \infty = -\infty$.
  - For any $a \in \mathbb{R}$, we have $a \leq \infty$. In addition, we also have $-\infty < \infty$.
  - For any $a \in \mathbb{R}$, we have $a \geq -\infty$. In addition, we also have $\infty > -\infty$.
  - $\infty - \infty$ is not defined, but $\infty + \infty = \infty$ and $-\infty - \infty = -\infty$.

  This thus allows ordering and addition properties of $\mathbb{R}$ to be partially preserved.
- If S is empty, then we denote $\sup S := -\infty$ and $\inf S = +\infty$.

Example 15. Consider the set S to be given by

$$S = \bigcup_{n=1}^{\infty} \left[-n, 1 - \frac{1}{n}\right]. \tag{23}$$

Compute (without justification), the following

(i) The set S itself, and hence

(ii) $\inf S$ and $\sup S$.

(iii) Does $\min S$ and $\max S$ exists?

Proof.

(i) Observe that at $n = 1$, we have $[-1, 0]$. At $n = 2$, we have $\left[-2, \frac{1}{2}\right]$. At $n = 3$, we have $\left[-3, \frac{1}{3}\right]$. We notice that by unioning them all up, we should obtain[9] $(-\infty, 1) := \{x \in \mathbb{R} : x < 1\}$. Here we are excluding 1 in our set since it is never attained by $\left[-n, 1 - \frac{1}{n}\right]$ for any natural number n. [10]

(ii) Using the set in (i), one can see that $\inf S = -\infty$ and $\sup S = 1$.

(iii) No for both.[11]

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[9] The notation for intervals are covered in Chapter 1 Section 5 of the textbook.

[10] More explictly, we recall the definition of $\bigcup_{n=1}^{\infty} S_n$ for sets S_n is that if $x \in \bigcup_{n=1}^{\infty} S_n$, then there exists $n \in \mathbb{N}$ such that $x \in S_n$. Thus, not being able to attain a value for every natural number excludes it from the set, while for any positive real number $y < 1$, this is attainable if we pick n sufficiently large such that $y < 1 - \frac{1}{n}$ (this is a consequence of Archimedean Property of $\mathbb{R}$ too, if one needs to prove this rigorously).

[11] Take $\max S$ for instance. If we claim that we have found a maximal element **in the set**, say 0.999, this is indeed not true as 0.9999 is larger. However, since 1 is not in the set, we can't pick 1 as the maximal element. This is the awkward "technicality" we fall into if we do not introduce $\sup$ (and similarly, $\inf$).

Here are some key properties/theorems/corollaries that you should keep in mind that will be useful for proofs about $\mathbb{R}$ in this entire course.

- Theorem 4.4, Completeness Axiom. Every non-empty subset of $\mathbb{R}$ that is bounded from above has a supremum (least upper bound).
- Corollary 4.5. Similarly, every non-empty subset of $\mathbb{R}$ that is bounded from below has an infimum (greatest lower bound).
- Theorem 4.6, Archimedean property of $\mathbb{R}$. For any real numbers $a, b > 0$, there exists a positive integer n such that $na > b$.
- Corollary. For any real number $a > 0$, there exists a positive integer n such that $a > \frac{1}{n}$. (Take $b = 1$ from above and divide by n throughout.)
- Theorem 4.7, Denseness of $\mathbb{Q}$ in $\mathbb{R}$. For any real numbers a, b such that $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Some additional remarks are as follows:

- More often than not, the corollary for Archimedean property of $\mathbb{R}$ is more important than the original property. We just want to rigorously say that if a real number is strictly greater than 0, we can always find a natural number n sufficiently large such that $a > 1/n$. Notationally, this is unintuitive. Intuitively, this is! Since $a > 0$ is given, then we know that it must be of some distance away from 0. Coupled with the fact that $\frac{1}{n}$ tends to 0 as n goes to infinity, this can then be achieved by picking n sufficiently large!

  Example: If $a = 10^{-2022}$. We just have to pick $n = 10^{2022} + 1$ and observe that $a = 10^{-2022} > \frac{1}{10^{2022}+1}$.
- Denseness of $\mathbb{Q}$ in $\mathbb{R}$ on the other hand is not as intuitive. It says that no matter how small the interval (a, b) is, as long as $b > a$ (ie there is some gap, and this gap $b - a > 0$), we can always find a rational number in this gap.
- In addition, since the denseness of $\mathbb{Q}$ in $\mathbb{R}$ relies on gap being greater than 0, you can see that we must invoke the corollary of Archimedean property of $\mathbb{R}$ somewhere in the proof!

The following are some proof-based questions on $\sup, \inf$ and the aforementioned properties.

Example 16. (Exercise 4.7, Modified.) Let S and T be subsets of $\mathbb{R}$.

(i) For non-empty **bounded** subsets S and T of $\mathbb{R}$, suppose that $S \subseteq T$. Prove that $\inf T \leq \inf S \leq \sup S \leq \sup T$.

(ii) For non-empty **(not necessarily bounded)** subsets S and T of $\mathbb{R}$, prove that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

(iii) Do the claims in (i) and (ii) hold if $S = T = \emptyset$?

Proof.

(i) Since S and T are non-empty and bounded, we know that $\sup S, \sup T$, $\inf S$, and $\inf T$ exist and are in $\mathbb{R}$ (ie not $\pm\infty$).

1. $\inf T \leq \inf S$.
   i. By definition of $\inf T$, we see that $\inf T \leq t$ for all $t \in T$.
   ii. Since $S \subseteq T$, then we see that $\inf T \leq s$ for all $s \in S$ (since if the bound holds for all T, the larger set, then it must hold for any subset of T).
   iii. Thus, observe that $\inf T$ is a lower bound of S.
   iv. By definition of $\inf S$ (as the greatest lower bound), we see that $\inf T \leq \inf S$.
2. $\inf S \leq \sup S$.
   i. This follows from the fact that for all $s \in S$, we have $\inf S \leq s$ and $s \leq \sup S$.
   ii. By transitive property of $\mathbb{R}$ (by picking any element say $y \in S$, and note that this element exists since S is non-empty), we see that $\inf S \leq \sup S$.
3. $\sup S \leq \sup T$.
   i. By definition of $\sup T$, we see that $t \leq \sup T$ for all $t \in T$.
   ii. Since $S \subseteq T$, then we see that $s \leq \sup T$ for all $s \in S$ (since if the bound holds for all T, the larger set, then it must hold for any subset of T).
   iii. Thus, observe that $\sup T$ is an upper bound of S.
   iv. By definition of $\sup S$ (as the lowest upper bound), we see that $\sup S \leq \sup T$.

(ii) Note that it is possible for S and T to be unbounded. We shall first deal with these cases in which $\sup S$ or $\sup T$ (or both) is equals to $+\infty$.

1. Suppose that either S or T is not bounded from above (or both).
   i. This implies that $S \cup T$ is not bounded from above, and hence $\sup(S \cup T) = +\infty$.
   ii. On the other hand, we know that either $\sup S = +\infty$ or $\sup T = +\infty$. In either case, we deduce that $\max\{\sup S, \sup T\} = +\infty$.
   iii. From i. and ii., we deduce that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.
2. We are now left with the case in which both S and T are bounded from above, and hence $\sup S < +\infty$ and $\sup T < +\infty$.
   i. By definition of $\sup$, we observe that $s \leq \sup S$ and $t \leq \sup T$ for all $s \in S$ and $t \in T$.
   ii. Now, take an element x in $S \cup T$. This implies that $x \in S$ or $x \in T$, which in turn implies that $x \leq \sup T$ or $x \leq \sup S$.
   iii. In both cases, we see that it is indeed true that $x \leq \max\{\sup T, \sup S\}$. (In the first case, $x \leq \sup T \leq \max\{\sup T, \sup S\}$; in the second case, $x \leq \sup S \leq \max\{\sup T, \sup S\}$.) Thus, $\max\{\sup T, \sup S\}$ is an upper bound for $S \cup T$.
   iv. Thus, by definition of $\sup(S \cup T)$ as the least upper bound for $S \cup T$, we have $\sup(S \cup T) \leq \max\{\sup S, \sup T\}$.
   v. On the other hand, we note from the proof in (i) that $S \subseteq T$ implies $\sup S \leq \sup T$ holds even if S and/or T are not bounded from below (they just have to be bounded from above so that $\sup S$ and $\sup T$ are not infinities, and the inequalities thus make sense).
   vi. Hence, by v., since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, we have $\sup S \leq \sup(S \cup T)$ and $\sup T \leq \sup(S \cup T)$.
   vii. The above thus implies that $\max\{\sup S, \sup T\} \leq \sup(S \cup T)$.

viii. Combining iv. and vii., this implies that $\max\{\sup S, \sup T\} = \sup(S \cup T)$, as required.

(iii) No, since by definition, $\sup S = -\infty$ and $\inf S = +\infty$, yet we still demand that $\inf S \leq \sup S$ (which is absurd).

Example 17. (Exercise 4.10.) Prove that if $a > 0$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

This question looks like the corollary of Archimedean property of $\mathbb{R}$, with an additional inequality $a > n$. The strategy here is to apply the Archimedean property of $\mathbb{R}$ (just like how the Corollary was obtained) to obtain the other inequality. As a recap (with a slight change in notation for disambiguation), for any real numbers $x, y > 0$, there exists a positive integer n such that $nx > y$.

Proof

1. By the Archimedean property of $\mathbb{R}$, there exists $n_1 \in \mathbb{N}$ such that $n_1 a > 1$. (Pick $x = a, y = 1$.) This implies that $a > \frac{1}{n_1}$.
2. By another application of Archimedean property of $\mathbb{R}$, there exists $n_2 \in \mathbb{N}$ such that $n_2 \cdot 1 > a$. (Pick $x = 1, y = a$.) This implies that $n_2 > a$.
3. To submit the existence of n, we pick $n = \max\{n_1, n_2\}$. We can then check that $a > \frac{1}{n_1} \geq \frac{1}{\max\{n_1,n_2\}} = \frac{1}{n}$ and $n = \max\{n_1, n_2\} \geq n_2 > a$. Then, we have $\frac{1}{n} < a < n$ as required.

Example 18. (Exercise 4.11.) Consider $a, b \in \mathbb{R}$, where $a < b$. Use the denseness of $\mathbb{Q}$ in $\mathbb{R}$ to show that there are infintely many rationals between a and b.

Proof.

1. By Theorem 4.7 (denseness of $\mathbb{Q}$ in $\mathbb{R}$), we see that there exists a rational number $r \in \mathbb{Q}$ such that $a < r < b$.
2. Next, observe that $b - r > 0$.
3. By the Archimedean property of $\mathbb{R}$, there exists a positive integer n such that $b - r > \frac{1}{n}$, and hence $r + \frac{1}{n} < b$.
4. We also know that $a < r$, so $a < r + \frac{1}{n} < b$. $r + \frac{1}{n}$ is now another rational number between a and b.
5. Next, observe that since $a < r$, so we also have
$$a < r + \frac{1}{n+1} < r + \frac{1}{n} < b.$$
Hence, $r + \frac{1}{n+1}$ is yet another rational number between a and b.
6. We can repeat the argument with $r + \frac{1}{n+k}$ for positive integer k to conclude that $r + \frac{1}{n+k}$ is also another rational number between a and b, for all natural numbers k.
7. 6. now implies that there are infinitely many rational numbers between a and b.

Limit of Sequences.

To begin, we shall introduce some key intuitive notations:

- Intuitively, a sequence is denoted by $(s_n)_{n=m}^{\infty}$, where m is the starting index. This is explicitly given by $(s_n)_{n=m}^{\infty} = (s_m, s_{m+1}, s_{m+2}, \cdots)$. If $m = 1$, we may write this as $(s_n)_{n\in\mathbb{N}}$. If the domain is clear, we might omit sub-scripting the index in which the sequence runs over.
- A sequence (s_n) of real numbers is said to **converge** to the real number s if

  $$\forall \varepsilon > 0, \quad \exists \text{ a real number } N, \quad n > N \implies |s_n - s| < \varepsilon.$$

  We write $\lim_{n\to\infty} s_n = s$, and denote s as the **limit** of the sequence (s_n).

  Intuitively, this means that if we demand that the sequence is to be close to the limit by some small error (that we are free to choose), we must be able to say how close into the sequence we should be in to be within the relevant error ("how close" here should depend on the given error).
- A sequence that does not converge to some real number is said to **diverge**.

We shall include an example of determining the limit of the sequence and prove that it is indeed the limit, and conclude with an example question on proving some properties of limits.

Example 19. (Exercise 8.2(a)) Determine the limit

$$\lim_{n\to\infty} \frac{n}{n^2+1}$$

and prove rigorously that the limit is indeed the value as mentioned above.

Proof.

1. **Intuition/Pre-Computation:** Observe that the limit is 0, since $n^2+1 \to n^2$ for large n, so $\frac{n}{n^2+1} \to \frac{n}{n^2} = \frac{1}{n} \to 0$. To prove that it is indeed true, we have to show that $\left|\frac{n}{n^2+1}\right| < \varepsilon$. For positive integers N and $n > N$, we have that both n and n^2 (and hence n^2+1) are positive.

2. Hence, we have
$$\left|\frac{n}{n^2+1}\right| < \frac{n}{n^2} = \frac{1}{n} < \varepsilon$$
if $\frac{1}{n} < \varepsilon$, or $n > \frac{1}{\varepsilon}$. Thus, it suffices to pick $N = \frac{1}{\varepsilon}$.

3. **Proof:** $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$. Let $\varepsilon > 0$ be given. Pick $N = \frac{1}{\varepsilon}$. Then, for all $n > N$ (and hence n is positive), we have
$$\left|\frac{n}{n^2+1}\right| < \frac{1}{n} < \varepsilon$$
by 2.

Remark: There are no set “formulas” that one can use to prove this; some of these limits might require going beyond solving an inequality (as seen from examples in the textbook), and requires a healthy dose of creativity!

Example 20. (Exercise 8.4.) Let (t_n) be a bounded sequence, i.e, there exists a constant $M > 0$ such that $|t_n| \leq M$ for all $n \in \mathbb{N}$, and let (s_n) be a sequence such that $\lim_{n\to\infty} s_n = 0$. Prove that $\lim_{n\to\infty} s_n t_n = 0$.

Proof.

1. To prove that the limit is 0, we have to show that

$$\forall \varepsilon > 0, \quad \exists \text{ a real number } N, \quad n > N \implies |s_n t_n - 0| = |s_n t_n| < \varepsilon.$$

2. We break down the hypothesis of the question, that is, we already know that the following holds:

$$\forall \varepsilon' > 0, \quad \exists \text{ a real number } N', \quad n > N' \implies |s_n - 0| = |s_n| < \varepsilon'.$$

3. To prove 1., we start by assuming that $\varepsilon > 0$ is given.

4. To pick the appropriate N, we observe that $|s_n t_n| \leq M|s_n|$ by using the assumption of the question ($|t_n| \leq M$ for all n). Hence, for $|s_n t_n| \leq \varepsilon$, we need $|s_n| \leq \frac{\varepsilon}{M}$.

5. Substitute $\varepsilon' = \frac{\varepsilon}{M}$ (since it is to be true for all ε', it must be true for $\frac{\varepsilon}{M}$). By 2., this implies that there exists a real number N' such that $n > N'$ implies that $|s_n| \leq \varepsilon' = \frac{\varepsilon}{M}$.

6. Now, to 1., we submit $N = N'$. Then, for all $n > N'$, we have that 2. holds ($|s_n| \leq \frac{\varepsilon}{M}$) and see that

$$\begin{aligned} |s_n t_n| &\leq M|s_n| \\ &\leq M\left(\frac{\varepsilon}{M}\right) \\ &= \varepsilon, \end{aligned} \tag{24}$$

as required. This concludes the proof.

3 Discussion 3

Limit Theorems, Monotone Sequences, and Cauchy Sequences.
Before we quote relevant limit theorems, let us first define what is mean for a sequence to have limits $\pm\infty$.

Definition 21. (Definition 9.8.) For a sequence (s_n),

- We write $\lim_{n\to\infty} s_n = +\infty$ provided for each $M > 0$ there is a number N such that $n > N$ implies that $s_n > M$. In this case, we say that the sequence diverges to $+\infty$.
- We write $\lim_{n\to\infty} s_n = -\infty$ provided for each $M < 0$ there is a number N such that $n > N$ implies that $s_n < M$. In this case, we say that the sequence diverges to $-\infty$.

The following are some limit theorems that you can use to compute relevant limits:

- Theorem 9.2. If the sequence (s_n) converges to s and $k \in \mathbb{R}$, then the sequence (ks_n) converges to ks. In other words,
$$\lim_{n\to\infty} (ks_n) = k \cdot \lim_{n\to\infty} s_n.$$
- Theorem 9.3. If (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)$ converges to $s + t$. In other words,
$$\lim_{n\to\infty} (s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n.$$
- Theorem 9.4. If (s_n) converges to s and (t_n) converges to t, then $(s_n t_n)$ converges to st. In other words,
$$\lim_{n\to\infty} (s_n t_n) = \lim_{n\to\infty} s_n \cdot \lim_{n\to\infty} t_n.$$
- Theorem 9.6. Suppose (s_n) converges to s and (t_n) converges to t. If $s \neq 0$ and $s_n \neq 0$ for all $n \in \mathbb{N}$, then $\left(\frac{t_n}{s_n}\right)$ converges to $\frac{t}{s}$. In other words,
$$\lim_{n\to\infty} \left(\frac{t_n}{s_n}\right) = \frac{\lim_{n\to\infty} t_n}{\lim_{n\to\infty} s_n}.$$
- Theorem 9.9. Let (s_n) and (t_n) be sequences such that $\lim_{n\to\infty} s_n = +\infty$ and $\lim_{n\to\infty} t_n > 0$ (which could be finite or $+\infty$). Then,
$$\lim_{n\to\infty} s_n t_n = +\infty.$$
- Theorem 9.10. Let (s_n) be a sequence of **positive** real numbers. Then,
$$\lim_{n\to\infty} s_n = +\infty \quad \text{if and only if} \quad \lim_{n\to\infty} \frac{1}{s_n} = 0.$$

The following theorem records common limits that one will encounter, in which when used in combination of the limit theorems above, can yield the value of limit (without going through the rigorous definition of $\varepsilon - N$ of a limit).

Theorem 22. (Theorem 9.7.)

- $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p > 0$.
- $\lim_{n\to\infty} a^n = 0$ if $|a| < 1$.
- $\lim_{n\to\infty} n^{1/n} = 1$.
- $\lim_{n\to\infty} (a^{1/n}) = 1$ for $a > 0$.

For explicit sequences (s_n) for $n \in \mathbb{N}$, we can usually use Theorem 9.7 with some combination of the limit theorems above to determine limits. For recurrence relations (say the formula for s_{n+1} is not explicitly given, but is given as a function of s_n and the previous terms), this might not be as clear that the sequence is indeed convergent. We will thus rely on relevant convergence theorems to prove that the limit exists in the first place.

Before we begin, let us flash out the relevant definitions for these sections in the textbook.

- (Section 9.) A sequence (s_n) of real numbers is said to be **bounded** if the set $\{s_n : n \in \mathbb{N}\}$ is a bounded set; if there exists a constant M such that $|s_n| \leq M$ for all n.
- (Definition 10.1.) A sequence (s_n) of real numbers is said to be **increasing** if $s_n \leq s_{n+1}$ for all n.
- (Definition 10.1.) A sequence (s_n) of real numbers is said to be **decreasing** if $s_n \geq s_{n+1}$ for all n.
- (Definition 10.1.) A sequence (s_n) of real numbers is said to be **monotone** if it is either increasing or decreasing.
- (Definition 10.8.) A sequence (s_n) of real numbers is said to be **Cauchy** if for each $\varepsilon > 0$ there exists a (real) number N such that $m, n > N$ implies

  $$|s_n - s_m| < \varepsilon.$$

  Intuitively, a Cauchy sequence is a sequence that gets arbitrarily close for large n.

Hence, we have the following (soft) properties of sequences:

- (Theorem 9.1.) Convergent sequences are bounded.
- (Theorem 10.2.) All bounded monotone sequences converge.
  (This is also known as the **Monotone Convergence Theorem.**)
- (Corollary 10.5.) If (s_n) is a monotone sequence, then the sequence either converges, diverges to $+\infty$, or diverges to $-\infty$. Thus, $\lim_{n\to\infty} s_n$ is always meaningful for monotone sequences.
- (Theorem 10.11.) A sequence is a convergent sequence if and only if it is a Cauchy sequence.

Thus, to prove that the limit of a sequence exists, we either

(i) Show that the sequence is

1. Bounded, and
2. Monotone,

OR

(ii) Show that the sequence is Cauchy.

Below are some examples of computing limits.

Example 23. Consider the sequence (s_n) for $n \in \mathbb{N}$ given by

$$s_1 = 2, \quad s_{n+1} = \sqrt{12 + s_n} \quad \text{for all } n \in \mathbb{N}. \tag{25}$$

Prove that (s_n) converges and find its limits.

Proof. A good start would be to list the first few values of the sequence. We see that $s_2 = \sqrt{12+2} = \sqrt{14} \approx 3.742$. The next few values of the sequence are $\sqrt{12+\sqrt{14}} \approx 3.967, \approx 3.996, \cdots$. From here. one can postulate that the sequence is increasing and bounded from above.

1. First, we prove that $s_n \leq 4$ for all $n \in \mathbb{N}$. We do so by induction.
   (i) Let P_n be the proposition "$s_n \leq 4$" for all $n \in \mathbb{N}$.
   (ii) Base Case: Prove that P_1 is true, ie $s_1 \leq 4$. Indeed, since $s_1 = 2$, then $s_1 = 2 \leq 4$.
   (iii) Induction Step: Assume that P_k is true for some $k \in \mathbb{N}$, ie $s_k \leq 4$. WTS that P_{k+1} is true, ie $s_{k+1} \leq 4$.
   (iv) Indeed, we have $s_{k+1} = \sqrt{12 + s_k} \leq \sqrt{12+4} = 4$, where we have used the induction hypothesis (P_k) for the inequality part. Thus, by induction, we deduce that $s_n \leq 4$ for all $n \in \mathbb{N}$.
2. Next, we prove that s_n is monotone. By the first few values of the sequence, we should expect that the sequence is increasing. We shall also do so by induction.
   (i) Let P_n be the proposition "$s_n \leq s_{n+1}$" for all $n \in \mathbb{N}$.
   (ii) Base Case: Prove that P_1 is true, ie $s_1 \leq s_2$. Indeed, since $s_1 = 2$ and $s_2 = \sqrt{14} \approx 3.742$, then $s_1 = 2 \leq \sqrt{14} = s_2$.
   (iii) Induction Step: Assume that P_k is true for some $k \in \mathbb{N}$, ie $s_k \leq s_{k+1}$. WTS that P_{k+1} is true, ie $s_{k+1} \leq s_{k+2}$.
   (iv) Indeed, we have $s_{k+2} = \sqrt{12 + s_{k+1}} \geq \sqrt{12 + s_k} = s_{k+1}$, where we have used the induction hypothesis (P_k) for the inequality part, and the recurrence relation for the last equality. Thus, by induction, we deduce that $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$.
3. Furthermore, from the recurrence relation, for the square root to be well defined, we must have $s_{n+1} \geq 0$ for all $n \in \mathbb{N}$ (or just $s_n \geq 0$ for all $n \in \mathbb{N}$). Since $0 \leq s_n \leq 4$ for all $n \in \mathbb{N}$, we say that the sequence is bounded. Thus, by Theorem 10.2, we deduce that the sequence (s_n) converges.
4. To determine the limit $\lim_{n\to\infty} s_n := s$, we see that as $n \to \infty$, $s_n \to s$ and $s_{n+1} \to s$. [12] Thus, going back to the recurrence relation in (25), we have

$$\begin{aligned} s_{n+1} &= \sqrt{12 + s_n} \\ \downarrow \quad & \quad\quad \downarrow \\ s &= \sqrt{12+s} \\ s^2 &= 12 + s \\ s^2 - s - 12 &= 0 \\ s &= -3 \quad \text{or} \quad 4. \end{aligned} \tag{26}$$

   Thus, we deduce that the limit must be either -3 or 4.
5. Using the fact that $0 \leq s_n \leq 4$, the limit must also satisfy this property. This implies that the limit must be 4. (This is consistent with what we obtain by listing the first few values of the sequence.)

[12] One could rigorously show that if $\lim_{n\to\infty} s_n = s$, then for any $k \in \mathbb{N}$ (translation of the sequence), we have $\lim_{n\to\infty} s_{n+k} = s$ (converging to the same limit).

Example 24. Let $s_n = \sqrt{n}$ for all $n \in \mathbb{N}$.

(i) Prove that $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$.

(ii) Is (s_n) a Cauchy sequence? Explain your answer.

Proof.

(i) First, we compute

$$\begin{aligned} |s_{n+1} - s_n| &= |\sqrt{n+1} - \sqrt{n}| \\ &= \frac{|\sqrt{n+1} + \sqrt{n}||\sqrt{n+1} - \sqrt{n}|}{|\sqrt{n+1} + \sqrt{n}|} \\ &= \frac{n+1-n}{|\sqrt{n+1} + \sqrt{n}|} \\ &= \frac{\frac{1}{\sqrt{n}}}{|\sqrt{1+\frac{1}{n}} + 1|} \\ &\to \frac{0}{2} = 0. \end{aligned} \tag{27}$$

Here, we use the fact that $(a-b)(a+b) = a^2 - b^2$ for any $a, b \in \mathbb{R}$, and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{2}}} = 0$ by Theorem 9.7 (with $p = \frac{1}{2}$), and $\lim_{n\to\infty} |\sqrt{1+\frac{1}{n}} + 1| = 2.$[13]

(ii) The sequence is not Cauchy.

1. Suppose for a contradiction that the sequence (s_n) above is Cauchy.
2. By Theorem 10.11, the sequence is convergent (in $\mathbb{R}$).
3. By Theorem 9.1, the sequence is bounded.
4. However, observe that since $\lim_{n\to\infty} \sqrt{n} = +\infty$, the sequence is unbounded and this contradicts 3.

Remark: Note that for (ii), one can make the contradiction between 3. and 4. more rigorous as follows.

1. Since (s_n) is bounded, there exists real number $M_1 > 0$ such that for all $n \in \mathbb{N}$, we have $|s_n| < M_1$.
2. However, since $\lim_{n\to\infty} \sqrt{n} = +\infty$. we have by the definition of limits that for every $M > 0$, there exists $N > 0$ such that for all $n > N$, we have $|s_n| > M$.
3. To the definition in 2, we set $M_1 = M$ from 1. This implies that $|s_n| < M$.
4. Thus, 2. (which says for $n > N$, we have $|s_n| > M$) contradicts 3. (which says that for all $n \in \mathbb{N}$, we have $|s_n| < M$).

[13] To be more rigorous, we would either have to use the $\varepsilon - N$ definition of a limit, or prove a couple of limit theroems; $\lim_{n\to\infty} \sqrt{s_n} = \sqrt{\lim_{n\to\infty} s_n}$ and $\lim_{n\to\infty} |s_n| = |\lim_{n\to\infty} s_n|$. The former is actually covered in lectures, so you might be able to cite that if needed.

Example 25. From Exercise 9.12 (b), it says that for all $s_n > 0$ and suppose that the limit $L = \lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right|$ exists, then if $L > 1$, then $\lim_{n\to\infty} s_n = +\infty$. Use this to prove that for any given real number $b > 1$ and natural number k,

$$\lim_{n\to\infty} \frac{b^n}{n^k} = +\infty.$$

Proof.

1. To apply the results from Exercise 9.12 (b), we have to first show that the limit $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right|$ exists and compute it.
2. Let $t_n = \left|\frac{s_{n+1}}{s_n}\right|$ for each $n \in \mathbb{N}$. This is now equivalent to showing that $\lim_{n\to\infty} t_n$ exists.
3. Compute $t_n = \left|\frac{b^{n+1}/(n+1)^k}{b^n/n^k}\right| = \frac{b^{n+1}}{b^n}\frac{n^k}{(n+1)^k} = b \cdot \left(\frac{1}{1+\frac{1}{n}}\right)^k$.
4. Using the relevant limit theorems, since $\frac{1}{n} \to 0$ as $n \to \infty$, we have that $\lim_{n\to\infty} t_n = b \cdot \left(\frac{1}{1+0}\right)^k = b > 1$.
5. Thus, we can now apply Exercise 9.12 (b), and deduce that $\lim_{n\to\infty} s_n = +\infty$.

Example 26. Let (s_n) be a Cauchy sequence in $\mathbb{R}$. Is the sequence (t_n) given by $t_n = s_n^2 - 3s_n + 1$ for all $n \in \mathbb{N}$, Cauchy? Explain your answer.

Proof.

1. The sequence (t_n) is Cauchy.
2. By Theorem 10.11, since the sequence (s_n) is Cauchy, it is convergent. We then denote its limit as $s := \lim_{n\to\infty} s_n$.
3. By using relevant limit theorems, we then deduce that the sequence (t_n) is convergent, with limit $t_n = s_n^2 - 3s_n + 1 \to s^2 - 3s + 1$.
4. By another application of Theorem 10.11, since the sequence (t_n) is convergent, it is Cauchy.

Subsequences, Limit Supremums, and Limit Infimums.

Before we begin, we shall first define the terms above:

Definition 27. Let (s_n) be a sequence in $\mathbb{R}$.

- (Definition 10.6.) $\limsup_{n\to\infty} s_n := \lim_{n\to\infty} \sup\{s_k : k > n\}$.
- (Definition 10.6.) $\liminf_{n\to\infty} s_n := \lim_{n\to\infty} \inf\{s_k : k > n\}$.
- (Definition 11.1.) A **subsequence** of (s_n) is a sequence $(t_k)_{k\in\mathbb{N}}$ in which $t_k = s_{n_k}$; where $n_k = \sigma(k)$ is such that $n_1 < n_2 < \cdots$, with σ known as the selection function to pick the corresponding index of the original sequence for the index on the subsequence.
- (Definition 11.6.) A **subsequential limit** is any real number or the symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of (s_n).

Intuitively, we are taking the $\sup$ and $\inf$ over large values of n only.

We start off by listing properties of $\limsup$ and $\liminf$ below. Let (s_n) be a sequence in $\mathbb{R}$.

- $\limsup$ and $\liminf$ always exists (as either a real number or $\pm\infty$).
- (Theorem 10.7, i.) If $\lim_{n\to\infty} s_n$ is defined (ie as a real number or $\pm\infty$), then $\liminf_{n\to\infty} s_n = \lim_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.
- (Theorem 10.7, ii.) If $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$, then the limit $\lim_{n\to\infty} s_n$ exists and is given by $\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.
- (Theorem 11.8.) Let S denote the set of subsequential limits. This is obtained by collating limits from convergent subsequences. Then,
  - (i) S is non-empty,
  - (ii) $\sup S = \limsup_{n\to\infty} s_n$, and $\inf S = \liminf_{n\to\infty} s_n$,
  - (iii) $\lim_{n\to\infty} s_n$ exists if and only if S has exactly one element, namely $\lim_{n\to\infty} s_n$.

On the other hand, here are some properties of subsequences of a sequence (s_n) in $\mathbb{R}$.

- (Theorem 11.3.) If a sequence (s_n) in $\mathbb{R}$ converges, then every subsequence converges to the same limit.
- (Theorem 11.4.) Every sequence (s_n) in $\mathbb{R}$ has a monotonic subsequence.
- (Theorem 11.5; Bolzano-Weierstrass Theorem.) Every bounded sequence has a convergent subsequence.
- (Theorem 11.7.) There exists a monotonic subsequence whose limit is $\limsup_{n\to\infty} s_n$ and there exists a monotonic subsequence whose limit is $\liminf_{n\to\infty} s_n$.

These properties will help in writing relevant proofs regarding subsequences. Nonetheless, let us start by looking at computational examples of subsequences (which is the focus for exercises in HW 3), and possibly append an example on proof-based properties of subsequences.

Example 28. (Exercise 11.3.) Consider the sequence $s_n = \cos\left(\frac{n\pi}{3}\right)$ for all $n \in \mathbb{N}$.

(i) Give an example of a monotonic subsequence.

(ii) Compute S, the set of subsequential limit(s) of (s_n).

(iii) Compute $\limsup_{n\to\infty} s_n$ and $\liminf_{n\to\infty} s_n$.

Proof. One should try to list down the first few values of the sequence to "observe a pattern". We have $\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \cdots$ and the pattern repeats itself after every 6 indices.

(i) By picking $n_k = 6k$ (ie the first value of the repeating pattern of length 6), we have $(s_{n_k}) = (1, 1, 1, 1, \cdots)$. This constant sequence is indeed monotonic (can be both decreasing and increasing by definition!).

(ii) $S = \left\{1, \frac{1}{2}, -\frac{1}{2}, -1\right\}$, by picking the corresponding constant subsequences (since the pattern repeats with length 6). [14]

(iii) By Theorem 11.8, we see that $\limsup_{n\to\infty} s_n = \sup S = 1$ and $\liminf_{n\to\infty} s_n = \inf S = -1$.

[14] Note that if we pick the constant subsequences, we can only determine that $\left\{1, \frac{1}{2}, -\frac{1}{2}, -1\right\} \subseteq S$, but not set equality (since these are some subsequential limits, but does not represent the values obtained from all possible subsequences). To show that $S \subset \left\{1, \frac{1}{2}, -\frac{1}{2}, -1\right\}$ rigorously, one has to argue that since the sequence only comprises of values 1, $\frac{1}{2}$, $-\frac{1}{2}$, and -1, then the subsequential limit must be either of these four.

Example 29. Let (s_n) be a sequence of real numbers.

(i) Prove that if both the even and odd subsequences (s_{2n}) and (s_{2n+1}) converges to the same limit, then the sequence (s_n) converges to the same limit

(ii) Use (i) to prove that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Proof.

(i) 1. Suppose that both the even and odd subsequences converges. Denote the limit as l. This implies that for every $\varepsilon > 0$, there exists real numbers N_1 and N_2 such that for all $n' > N_1$, we have $|s_{2n'} - l| < \varepsilon$ and for all $n' > N_2$, we have $|s_{2n'+1} - l| < \varepsilon$.

2. Claim: $\lim_{n\to\infty} s_n = l$.

3. Let $\varepsilon > 0$ be given. We want to show that there exists an N such that for all $n > N$, we have $|s_n - l| < \varepsilon$.

4. Pick $N = \max\{2N_1, 2N_2 + 1\}$. Then $n > N$ implies that $n > 2N_1$ and $n > 2N_2$. If n is even, then $n = 2n'$ for some $n' \in \mathbb{N}$, so $n = 2n' > 2N_1$ implies that $n' > N_1$, and thus $|s_{2n'} - l| = |s_n - l| < \varepsilon$. If n is odd, then $n = 2n' + 1$ for some $n' \in \mathbb{N}$, so $n = 2n' + 1 > 2N_2 + 1$ implies that $n' > N_2$, and thus $|s_{2n'+1} - l| = |s_n - l| < \varepsilon$. All in all, we have for all $n > N$, $|s_n - l| < \varepsilon$, and we are done.

(ii) 1. Consider the even subsequence, and see that $s_{2n} = \frac{(-1)^{2n}}{2n} = \frac{1}{2n}$, and thus $\lim_{n\to\infty} \frac{1}{2n} = 0$. Similarly, for the odd subsequence, we see that $s_{2n+1} = \frac{(-1)^{2n+1}}{2n+1} = -\frac{1}{2n+1}$, and thus $\lim_{n\to\infty} -\frac{1}{2n+1} = 0$.

2. Since both even and odd subsequences converge to the same limit, we have that the limit of the original sequence exists and is equals to this limit. Thus, $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

4 Discussion 4

$\limsup$ and $\liminf$.

Recall from Definition 10.6, we have

Definition 30. Let (s_n) be a sequence in $\mathbb{R}$.

- (Definition 10.6.) $\limsup_{n\to\infty} s_n := \lim_{n\to\infty} \sup\{s_k : k > n\}$.
- (Definition 10.6.) $\liminf_{n\to\infty} s_n := \lim_{n\to\infty} \inf\{s_k : k > n\}$.

Furthermore, from the previous Discussion, we reviewed the following properties of $\limsup$ and $\liminf$.

- $\limsup$ and $\liminf$ always exists (as either a real number or $\pm\infty$), since the sequences $\sup\{s_k : k > n\}$ and $\inf\{s_k : k > n\}$ are monotonic, so the limit exists in $\mathbb{R} \cup \{\pm\infty\}$ (recall that infinities are excluded if the given sequence is bounded).
- (Theorem 10.7, i.) If $\lim_{n\to\infty} s_n$ is defined (ie as a real number or $\pm\infty$), then $\liminf_{n\to\infty} s_n = \lim_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.
- (Theorem 10.7, ii.) If $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$, then the limit $\lim_{n\to\infty} s_n$ exists and is given by $\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.
- (Theorem 11.7.) There exists a monotonic subsequence whose limit is $\limsup_{n\to\infty} s_n$ and there exists a monotonic subsequence whose limit is $\liminf_{n\to\infty} s_n$.
- (Theorem 11.8.) Let S denote the set of subsequential limits. This is obtained by collating limits from convergent subsequences. Then,
  - (i) S is non-empty,
  - (ii) $\sup S = \limsup_{n\to\infty} s_n$, and $\inf S = \liminf_{n\to\infty} s_n$,
  - (iii) $\lim_{n\to\infty} s_n$ exists if and only if S has exactly one element, namely $\lim_{n\to\infty} s_n$.

Before we look at some proof-based problems on $\limsup$ and $\liminf$, in my personal opinion, proving the properties by Definition 10.6 is one of the easiest method to prove $\limsup$ and $\liminf$ properties! This can be done by translating the properties to just properties of $\sup\{s_k : k > n\}$ (which are properties that most of us are (hopefully) more comfortable with), and then take the relevant limit. Here, we note that the limit always exists since the sequence $s_n = \sup\{s_k : k > n\}$ is increasing (monotone) and thus always exists (in $\mathbb{R} \cup \{\pm\infty\}$). We shall see this in action in the following two exercises.

Example 31. Let (s_n) and (t_n) be bounded sequences in $\mathbb{R}$, for $n \in \mathbb{N}$.

(i) Let I be some non-empty subset of $\mathbb{N}$. Prove that

$$\sup\{s_n + t_n : n \in I\} \leq \sup\{s_n : n \in I\} + \sup\{t_n : n \in I\}.$$

(ii) Hence or otherwise, prove that

$$\limsup_{n\to\infty}(s_n + t_n) \leq \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n.$$

(iii) Give an example of a pair of sequences (s_n) and (t_n) in $\mathbb{R}$ such that equality in (ii) does not hold; ie,

$$\limsup_{n\to\infty}(s_n + t_n) < \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n.$$

Proof.

(i) 1. Let n be an arbitrary element in I. Then, we have

$$\begin{aligned} s_n + t_n &\leq s_n + \sup\{t_n : n \in I\} \\ &\leq \sup\{s_n : n \in I\} + \sup\{t_n : n \in I\} \end{aligned}$$

since $s_n \leq \sup\{s_n : n \in I\}$ for each $n \in I$ and similarly, $t_n \leq \sup\{t_n : n \in I\}$ for each $n \in I$.

2. The upper bound in 1. is now independent of n. Since $s_n + t_n \leq$ "upper bound in 1." for all $n \in I$ (and the upper bound does not change with varying n), then we must have that $\sup\{s_n + t_n : n \in I\} \leq$ "upper bound in 1."
(In other words, the largest element among the indices $n \in I$ must also be bounded from above by the same value!)

3. Thus, we have

$$\sup\{s_n + t_n : n \in I\} \leq \sup\{s_n : n \in I\} + \sup\{t_n : n \in I\}.$$

(ii) 1. First, since $\sup\{s_k : k > n\}$, $\sup\{t_k : k > n\}$, and $\sup\{s_k + t_k : k > n\}$ are increasing (and bounded by the question) sequences in $\mathbb{R}$, by Theorem 10.2 (or Monotone Convergence Theorem), we have that each of these sequences converges (and thus the individual limits exists).

2. By changing the indices from n to k, and setting $k \in I$ if $k > n$ for any given $n \in \mathbb{N}$, we have that

$$\sup\{s_k + t_k : k > n\} \leq \sup\{s_k : k > n\} + \sup\{t_k : k > n\}.$$

3. Using relevant limit theorems (say $x_n \leq y_n$ implies $\lim x_n \leq \lim y_n$ and $\lim(x_n + y_n) = \lim x_n + \lim y_n$ if the sequences (x_n) and (y_n) converge), we "take limits on both sides of (i)" as $n \to \infty$, and obtain

$$\begin{aligned} \lim_{n\to\infty} \sup\{s_k + t_k : k > n\} &\leq \lim_{n\to\infty} \sup\{s_k : k > n\} + \lim_{n\to\infty} \sup\{t_k : k > n\} \\ \limsup_{n\to\infty}(s_n + t_n) &\leq \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n. \end{aligned}$$

(iii) Consider $(s_n) = ((-1)^n) = (-1, 1, -1, 1, -1, 1, \cdots)$ and $(t_n) = ((-1)^{n+1}) = (1, -1, 1, -1, 1, -1, \cdots)$. We can see that $\limsup s_n = 1$ and $\limsup t_n = 1$, yet since $s_n + t_n = 0$ for all $n \in \mathbb{N}$, we have

$$\limsup(s_n + t_n) = 0 < 2 = 1 + 1 = \limsup s_n + \limsup t_n.$$

Example 32.

(i) (Exercise 11.8.) Let (s_n) be a bounded sequence in $\mathbb{R}$. Prove that

$$\liminf_{n\to\infty}(-s_n) = -\limsup_{n\to\infty} s_n.$$

(ii) Let (t_n) be a bounded sequence in $\mathbb{R}$ such that for any bounded sequence (u_n) in $\mathbb{R}$, we have

$$\liminf_{n\to\infty}(t_n+u_n) = \liminf_{n\to\infty}(t_n) + \liminf_{n\to\infty}(u_n).$$

Prove that the sequence (t_n) is Cauchy.

Proof

(i) 1. Recall from Exercise 5.4 that we have $\sup(-S) = -\inf(S)$ for any nonempty subset S of $\mathbb{R}$. Set $S = \{-s_k : k > n\}$ for each given $n \in \mathbb{N}$, This implies that

$$\sup\{s_k : k > n\} = -\inf\{-s_k : k > n\}.$$

2. Since the sequences $\sup\{-s_k : k > n\}$ and $\inf\{s_k : k > n\}$ are monotone (increasing and decreasing respectively) and bounded (as given in the question), their corresponding limits exist in $\mathbb{R}$.

3. Using relevant limit theorems (here, $\lim(-x_n) = -\lim x_n$ if (x_n) is a convergent sequence), we have

$$\begin{aligned}\lim_{n\to\infty}\inf\{-s_k : k > n\} &= -\lim_{n\to\infty}\sup\{s_k : k > n\}\\ \liminf_{n\to\infty}(-s_n) &= -\limsup_{n\to\infty} s_n,\end{aligned}$$

as required.

(ii) 1. For any given bounded sequence (t_n), since the aforementioned property is true for any bounded sequence (u_n), we can pick $u_n = -t_n$ for each $n \in \mathbb{N}$.

2. With the help of (i), this thus implies that

$$\begin{aligned}\liminf_{n\to\infty}(t_n + (-t_n)) &= \liminf_{n\to\infty}(t_n) + \liminf_{n\to\infty}(-t_n)\\ \liminf_{n\to\infty}(0) &= \liminf_{n\to\infty}(t_n) - \limsup_{n\to\infty}(t_n)\\ \limsup_{n\to\infty}(t_n) &= \liminf_{n\to\infty}(t_n).\end{aligned}$$

We have used (i) in the second line.

3. By Theorem 10.7, we then deduce that the sequence (t_n) converges, with limit equals to $\limsup_{n\to\infty}(t_n) = \liminf_{n\to\infty}(t_n)$.

4. Since the sequence (t_n) is convergent, then it is Cauchy in $\mathbb{R}$. (By Theorem 10.11.)

Series and Common Convergent Tests.
Description/Terminologies associated with a series:

- For any integer $m < n$, $\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$. (Summation Notation.)
- The infinite series $\sum_{k=m}^{\infty} a_k$ means $\lim_{n\to\infty} \sum_{k=m}^{n} a_k$. Thus, it only makes sense to write the infinite series if the sequence $(\sum_{k=m}^{n} a_k)_n$ is convergent.
- Here, for a given sequence (a_n), the sequences of **partial sums** are denoted as $s_n = \sum_{k=m}^{n} a_k$.
- When the context is clear, we write $\sum a_n$ to represent the infinite sum $\sum_{k=m}^{\infty} a_k$.
- We say that a series $\sum_{k=m}^{\infty} a_k$ is **convergent** if the limit $\lim_{n\to\infty} \sum_{k=m}^{n} a_k$ exists.
- We say that a series $\sum_{k=m}^{\infty} a_k$ is **absolutely convergent** if the limit $\lim_{n\to\infty} \sum_{k=m}^{n} |a_k|$ exists.
- (Definition 14.3.) We say a series $\sum a_n$ satisfies the **Cauchy criterion** if its sequence of partial sums (s_n) is a Cauchy sequence; for every $\varepsilon > 0$, there exists a number N such that for all $n \geq m > N$,[15] we have[16] $|s_n - s_{m-1}| = |\sum_{k=m}^{n} a_k| < \varepsilon$.

Some (soft) properties of series include:

- (Theorem 14.4) A series converges if and only if it satisfies the Cauchy criterion.
- (Corollary 14.5.) If a series $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
- (Corollary 14.7.) Absolutely convergent series are convergent.

Coming to the main part of this section, the key is to use the correct test for convergence of a given infinite series. These can be summarized below.

- (Theorem 14.6, Comparison Test.) Let $\sum a_n$ be a series where $a_n \geq 0$ for all n.
  (i) If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges.
  (ii) If $\sum a_n = +\infty$ and $b_n \geq a_n$ for all n, then $\sum b_n = +\infty$.
- (Theorem 14.8, Ratio Test.) A series $\sum a_n$ of nonzero terms
  1. Converges absolutely if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$,
  2. Diverges if $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$.
  3. Otherwise, $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$, and the test gives no information.

  (Remark: If the limit $\left|\frac{a_{n+1}}{a_n}\right|$ exists, we can just replace both $\liminf$ and $\limsup$ above by $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ and compute this value, which is also known as the **ratio**, since it tells us the factor in which the next term in the sequence will "grow" in comparison with the previous term.)
- (Theorem 14.9, Root Test.) Let $\sum a_n$ be a series, and let $\alpha = \limsup |a_n|^{1/n}$. The series
  1. Converges absolutely if $\alpha < 1$, and
  2. Diverges if $\alpha > 1$.
  3. Otherwise $\alpha = 1$ and the test gives no information.

  (Remark: Root Test is always more powerful than the Ratio Test, with the only drawback that it would be hard to compute the value $\alpha = \limsup |a_n|^{1/n}$.)
- (Integral Test.) Let $\sum a_n$ where $a_n \geq 0$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, if we can find an integral such that

  $$\sum_{k=1}^{n} a_k \leq \text{ some integral} < +\infty \quad (\text{independent of } n),$$

  then the sum on the left remains bounded for all $n \in \mathbb{N}$. It is now a bounded monotone sequence of partial sums, and thus converges (with limit in $\mathbb{R}$). We can use a similar idea in testing for divergence.

[15] Here, without loss of generality, we set $n \geq m$.
[16] $s_n - s_m$ and $s_n - s_{m-1}$ is just a notational difference, so it doesn't matter which one we pick here.

- (Theorem 15.3, Alternating Series Test.) If a_n is a decreasing sequence with each $a_n \geq 0$ such that $a_n \to 0$ as $n \to \infty$, then the alternating series $\sum (-1)^{n+1} a_n$ converges. Moreover, the partial sums $s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$ satisfies $|s - s_n| \leq a_n$ for all n.[17]

With that, let us look at some common series which we can use for "comparison" (literally, for the comparison test above).

- Geometric Series $\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$. Thus, we can take $\lim_{n\to\infty}$ on both sides of the equality and the series therefore converges in $\mathbb{R}$ if $|r| < 1$.
- $p-$series (Theorem 15.1.)
$$\sum \frac{1}{n^p} \text{ converges if and only if } p > 1.$$
- Method of Differences. If a series has the form
$$s_n = \sum_{k=1}^{n} (a_k - a_{k-1}),$$
then we can evaluate the partial sums s_n as follows
$$\begin{aligned} s_n &= \sum_{k=1}^{n} (a_k - a_{k-1}) \\ &= \cancel{a_1} - a_0 \\ &+ \cancel{a_2} - \cancel{a_1} \\ &+ \cancel{a_3} - \cancel{a_2} \\ &\vdots \\ &+ \cancel{a_{n-1}} - \cancel{a_{n-2}} \\ &+ a_n - \cancel{a_{n-1}} \\ &= a_n - a_0. \end{aligned}$$
Thus, the convergence of the series is equivalent to the convergence of the sequence a_n.

In summary, for a series $\sum a_n$ with partial sums s_n,

Test for convergence	Test for divergence
Comparison Test (Theorem 14.6,(i)).	Comparison Test (Theorem 14.6,(ii)).
Ratio/Root Tests.	Ratio/Root Tests.
Integral Tests.	Integral Tests.
Alternating Series Test.	N.A.
Looking at absolute values; Corollary 14.7.	N.A
N.A,	$a_k \not\to 0$; Contrapositive, Corollary 14.5.
Evaluate series, s_n converges	Evaluate series, s_n diverges.

We shall look at a couple of computational examples, which will be the focus for this discussion.

[17] It does not matter if it is $\sum(-1)^{n+1}a_n$ or $\sum(-1)^n a_n$. Take out the minus sign, show the result series converges, and use relevant limit theorems to interchange the negative sign with taking limits!

Example 33. (Mixture of Exercises 14.1, 14.3, and others.) Determine which of the following series converge.
Difficulty: Easy - Medium.

(i) $\sum \frac{n^4}{2^n}$.

(ii) $\sum \frac{2^n}{n!}$.

(iii) $\sum \frac{2+\cos(n)}{3^n}$.

(iv) $\sum (-1)^n \log(n)$.

(v) $\sum \frac{1}{\log(n+1)}$.

(vi) $\sum \frac{(-1)^n}{\sqrt{n^2+3}}$.

(i) **Intuition:** 2^n grows faster than n^4, and thus is the "dominant" term, so it should converge.
Proof:

1. Compute the sequence $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+1}{n}\right|^4 \left|\frac{2^n}{2^{n+1}}\right| = \frac{1}{2}\left|\frac{n+1}{n}\right|^4$.
2. The limit of this sequence exists, and is equals to $\frac{1}{2}$ as $n \to \infty$.
3. By Ratio Test, since this limit, equals to $\limsup\left|\frac{a_{n+1}}{a_n}\right|$, is $\frac{1}{2} < 1$, then the series converges absolutely, and thus the series converges (by Corollary 14.7).

(ii) **Intuition:** For large n, the numerator is increased by a factor of 2, while the denominator is increased by a factor of n, and thus the "ratio" is given by $\frac{2}{n}$, which goes to zero. This implies that the series converges (and signals that we should use the Ratio Test).
Proof:

1. Compute the sequence $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}}{2^n}\right|\left|\frac{n!}{(n+1)!}\right| = \frac{2}{n}$.
2. The limit of this sequence exists, and is equals to 0 as $n \to \infty$.
3. By Ratio Test, since this limit, equals to $\limsup\left|\frac{a_{n+1}}{a_n}\right|$, is $0 < 1$, then the series converges absolutely, and thus the series converges (by Corollary 14.7).

(iii) **Intuition:** A good fact to know is that the $\cos$ term is always oscillating between -1 and 1 (and thus bounded between -1 and 1). Thus, it contributes to essentially a constant in the numerator, and the series is thus essentially $\frac{2+1}{3^n}$, a geometric series that converges.
Proof:

1. See that $|b_n| = \left|\frac{2+\cos(n)}{3^n}\right| \le \frac{3}{3^n} = a_n$, and that the series $\sum a_n = \sum \frac{3}{3^n}$ is a geometric series with common ratio $\frac{1}{3} < 1$ and thus converges.
2. By the Comparison test, this implies that $\sum b_n = \sum \frac{2+\cos(n)}{3^n}$ converges.

(iv) **Intuition:** If the term that we add on to the series for large n is not going to 0, then we can see that the series "does not stabilize" to a value. This is essentially the contrapositive of Corollary 14.5, which says that if $a_n \nrightarrow 0$, then the series $\sum a_n$ diverges. Indeed, since $(-1)^n \log(n)$ does not even converges (ie will oscillates between $+\infty$ and $-\infty$), then we expect that the series diverge.
Proof:

1. Observe that the sequence $a_n = (-1)^n \log(n)$ does not even converge in the first place. [18]
2. By the contrapositive of Corollary 14.5, the series diverges.

[18] To be rigorous, we can prove this. Suppose for a contradiction that it converges to a limit in $\mathbb{R}$, then all subsequences converges to the same limit in $\mathbb{R}$. Pick the even subsequence $a_{2n} = (-1)^{2n}\log(2n) = \log(2n)$, and observe that it goes to $+\infty$ as $n \to \infty$, while it should have been a value in $\mathbb{R}$ - a contradiction.

(v) **Intuition:** We note that the growth of $\log(n+1)$ is slower than n, so $\frac{1}{\log(n+1)}$ is bounded from above by $\frac{1}{n}$. This indicates the use of comparison test to prove divergence.
Proof:

1. Claim: For all $n \in \mathbb{N}$, $n \geq \log(n+1)$ (and thus $\frac{1}{\log(n+1)} \geq \frac{1}{n}$). We shall prove this by induction.
   Let P_n be the proposition "$n \geq \log(n+1)$" for all $n \in \mathbb{N}$.
   Base Case. P_1 is true since $1 \geq \log(2) \approx 0.302$.
   Induction Step. Assume that P_n is true for some $n \in \mathbb{N}$, that is, $n \geq \log(n+1)$. We want to show that $n+1 \geq \log(n+2)$ (P_{n+1} is true).

   Indeed, we have $n+1 \geq \log(n+1)+1$ by the induction hypothesis. It remains to show that $\log(n+1)+1 \geq \log(n+2)$ for each $n \in \mathbb{N}$. This is equivalent to showing that $\log(n+1)-\log(n+2) \geq -1$, or $\log\left(\frac{n+1}{n+2}\right) \geq -1$, or $\frac{n+1}{n+2} = 1 - \frac{1}{n+2} \geq e^{-1}$. Equivalently, we have to show that $\frac{1}{n+2} \leq 1-e^{-1}$, or $n+2 \geq \frac{1}{1-e^{-1}}$, or just $n \geq \frac{1}{1-e^{-1}} - 2 \approx -0.418$, which is true!
2. Thus, see that $|b_n| = \left|\frac{1}{\log(n+1)}\right| \leq \frac{1}{n} = a_n$, and $\sum a_n$ diverges ($p-$series with $p=1$).
3. By the Comparison Test, we deduce that $\sum b_n = \sum \frac{1}{\log(n+1)}$ diverges.

(vi) **Intuition:** With the $(-1)^n$ term, we should think of alternating series test if possible. Indeed, observe that without $(-1)^n$, the terms $a_n = \frac{1}{\sqrt{n^2+3}}$ is decreasing with increasing n (since the denominator gets larger), and are non-negative for each n.
Proof:

1. Observe that for an alternating series of the form $\sum(-1)^n a_n$, we have $a_n = \frac{1}{\sqrt{n^2+3}}$, with a_n decreasing with increasing n and $a_n \geq 0$ for each $n \in \mathbb{N}$.
2. By the alternating series test, the series $\sum \frac{(-1)^n}{\sqrt{n^2+3}}$ converges.

Example 34. Determine which of the following series converge.
Difficulty: Medium - Somewhat Challenging.

(i) $\sum (-1)^n \frac{1+\sin(n)}{3^n}$.

(ii) $\sum \frac{3^{2n}}{5^n}\left(1-\frac{1}{4n}\right)^{n^2}$. Hint: $\lim_{n\to\infty}(1-\frac{1}{n})^n = \frac{1}{e}$.

(iii) $\sum (-1)^n \left(\frac{2+3n}{3+4n}\right)^n$.

(iv) $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\sqrt{n^3-n}}$.

(i) **Intuition:** $\sin(n)$ is bounded between -1 and 1, so this is essentially $\sum \frac{2}{3^n}$. By using the bound on $\sin(n)$, we have to take absolute sign here so $(-1)^n$ disappears. This is then essentially a geometric series with common ratio $r = \frac{1}{3} < 1$, so it converges (by Comparison Test).
Proof:

1. See that $|b_n| = \left|(-1)^n \frac{1+\sin(n)}{3^n}\right| \leq \frac{2}{3^n} = a_n$, and that the series $\sum a_n = \sum \frac{2}{3^n}$ is a geometric series with common ratio $\frac{1}{3} < 1$ and thus converges.
2. By the Comparison test, this implies that $\sum b_n = \sum (-1)^n \frac{1+\sin(n)}{3^n}$ converges.

(ii) **Intuition:** This series looks like an absolute horror, with powers of n. However, powers of n gets removed when we take powers of $\frac{1}{n}$, and this is basically how the root test works. Thus, we shall hope that Root Test works and pray for the best.
Proof:

1. Compute the sequence $|a_n|^{1/n} = \left|\frac{9}{5}\right|(1-\frac{1}{4n})^n$.
2. The limit of this sequence exists, and can be computed as follows. Observe that

$$\left(1-\frac{1}{4n}\right)^n = \left(\left(1-\frac{1}{4n}\right)^{4n}\right)^{\frac{1}{4}}$$

and $\lim_{n\to\infty}\left(1-\frac{1}{4n}\right)^{4n} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e}$.[19] Hence, the limit $|a_n|^{1/n}$ as $n \to \infty$ exists and is equals to $\frac{9}{5}\left(\frac{1}{e}\right)^{\frac{1}{4}} \approx 1.40 > 1$.

3. By Root Test, since this limit, equals to $\limsup |a_n|^{1/n}$, is approximately $1.40 > 1$, then the series diverges.

(iii) **Intuition:** Similar to (ii), with the power of n present, your best bet (and friend, well, at least for this class) is the Root Test.
Proof:

1. Compute the sequence $|a_n|^{1/n} = \left|\frac{2+3n}{3+4n}\right|$.
2. The limit of this sequence exists, and is given by $\frac{3}{4}$.
3. By Root Test, since this limit, equals to $\limsup |a_n|^{1/n}$, is approximately $\frac{3}{4} < 1$, then the series converges absolutely, and thus converges (by Corollary 14.7).

(iv) **Intuition:** For this question, $(-1)^n$ is no longer a red herring (ie doing nothing!). In fact, if we take modulus here, for large n, the correct behavior is that we have $\frac{n}{\sqrt{n^3}} = \frac{1}{n^{1/2}}$, the term of a $p-$series with $p = \frac{1}{2}$ and thus diverges. Thus, we have to appeal to Alternating Series Test. To do so, we will have to show that $\frac{n}{\sqrt{n^3-n}}$ is decreasing for $n \geq 2$.
Proof:

1. Observe that for an alternating series of the form $\sum (-1)^n a_n$, we have $a_n = \frac{n}{\sqrt{n^3-n}}$. It is also clear that $a_n \geq 0$ for each $n \in \mathbb{N}$.

[19] Justification: We know that $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(1-\frac{1}{n})^n = \frac{1}{e}$, so all subsequences converges to the same limit. In particular, the subsequence (a_{4n}) also converges to $\frac{1}{e}$, and hence $\lim_{n\to\infty}\left(1-\frac{1}{4n}\right)^{4n} = \frac{1}{e}$.

2. Claim: a_n is decreasing for $n \geq 2$. Instead of induction, we shall provide a direct proof:

$$\begin{aligned} a_{n+1} &= \frac{n+1}{\sqrt{(n+1)^3-(n+1)}} \\ &= \frac{1}{\sqrt{n+1-\frac{1}{n+1}}} \\ &\geq \frac{1}{\sqrt{n-\frac{1}{n}}} \\ &= a_n. \end{aligned}$$

The inequality follows from the fact that the denominator is smaller; $n - \frac{1}{n} \leq n+1-\frac{1}{n+1}$ for any $n \in \mathbb{N}$.

3. By the alternating series test, the series $\sum(-1)^n a_n = \sum_{n=2}^{\infty}(-1)^n \frac{n}{\sqrt{n^3-n}}$ converges.

Example 35. Give an example of a convergent series $\sum a_n$ such that $\sum a_n^{\frac{1}{\pi}}$ diverges but $\sum a_n^{\frac{1}{e}}$ converges.

Proof.

1. **Intuition:** Use $p-$series. We will try to manipulate the choice of p so that we get convergence with the correct powers in accordance to Theorem 15.1.
2. **Proof:** Consider the series $\sum \frac{1}{n^p}$. It converges if $p > 1$, and this will be the range of p that we pick.
3. Now, we demand that $\sum \left(\frac{1}{n^p}\right)^{1/e} = \sum \frac{1}{n^{p/e}}$ converges. This implies that we need $\frac{p}{e} > 1$, or in other words, $p > e$.
4. Furthermore, we demand that $\sum \left(\frac{1}{n^p}\right)^{1/\pi} = \sum \frac{1}{n^{p/\pi}}$ diverges. This implies that we must have $\frac{p}{\pi} \leq 1$, ie $p < \pi$.
5. Combining 2., 3., and 4., we just have to pick a p such that $1 < e < p < \pi$. An appropriate choice would be $p = 3$. Thus, $\sum \frac{1}{n^3}$ is the required example.

5 Discussion 5

Functions, Continuity, and Uniform Continuity - Definitions and Computational/Intuitive Approach.

Recall the defining properties of a real-valued function (ie its co-domain/output space is $\mathbb{R}$), $f : \text{dom}(f) \to \mathbb{R}$:

- $\text{dom}(f)$ refers to the **domain** of f. This is assumed to be a subset of $\mathbb{R}$ for this class.
- For f to be a well-defined function, for each $x \in \text{dom}(f)$, there exists a unique element in (the co-domain/target set) $\mathbb{R}$, say y, such that $f(x) = y$.
(That is, we assign the output of f with the input x as y, and this must be in the target set.)
- In this class, if the domain is unspecified, then it must be the natural domain.
(That is, the largest possible subset of $\mathbb{R}$ as the domain such that $f(x)$ is well defined for each element in the domain; i.e $\frac{1}{x}$ has natural domain $\mathbb{R} \setminus \{0\}$.)

Here are two important definitions for **continuity** of a function:

Definition 36. We say that a function $f : \text{dom}(f) \to \mathbb{R}$ is **continuous** at some $x_0 \in \text{dom}(f)$ if and only if either of the criteria (definition) is satisfied

1. (Definition 17.1.) For every sequence (x_n) in $\text{dom}(f)$ converging to some $x_0 \in \text{dom}(f)$, we have that $\lim_{n\to\infty} f(x_n) = f(x_0)$.
2. (Theorem 17.2. $\varepsilon - \delta$ definition.)

$$\forall\, \varepsilon > 0, \exists\, \delta > 0, \text{ such that } \forall x \in \text{dom}(f), (|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon).$$

Remark: In some books, the first definition is known as the **sequential criterion of continuity** (since it is defined using sequences).

Additional Remark: The second definition above is also intuitive; it says that for a given distance ε, how close (ie tell me a δ) must my input x be to x_0 such that the difference in their individual outputs $|f(x) - f(x_0)|$ is smaller than the imposed distance ε. If the function is continuous, no matter how small we demand the imposed distance to be, we must always be able to shrink our distance of input x to x_0 sufficiently small to achieve this distance!

Yet An Additional Remark: We say that a function is continuous on a subset of $\text{dom}(f)$ if it is continuous at every point in this subset.

To prove that a given function is continuous, one can rely on Definition 36 or make use of the following Theorems and Facts:

Theorem 37. Let f and g be real-valued functions that are continuous at x_0 in $\mathbb{R}$. The following properties hold:

- (Theorem 17.3.) $|f|$ and kf (for $k \in \mathbb{R}$) are continuous at x_0.
- (Theorem 17.4.) $f + g$ is continuous at x_0.
- (Theorem 17.4.) fg is continuous at x_0.
- (Theorem 17.4.) f/g is continuous at x_0 if $g(x_0) \neq 0$.

On the other hand, if f is continuous at x_0 and g is continuous at $f(x_0)$, then the composition function $g \circ f$ is continuous at x_0. Furthermore, the following functions are continuous (Exercise 17.3):

- $\sin(x)$, $\cos(x)$, e^x, 2^x.
- $\log_e x$ and x^p for $x > 0$ (here, p can be any real number.)

We will see example(s) on these in a bit.

Next we introduce the concept of **uniform continuity**.

Definition 38. Let f be a real-valued function defined on a set $S \subseteq \mathbb{R}$. Then f is **uniformly continuous** on S if

$$\forall\, \varepsilon > 0, \exists\, \delta > 0, \text{ such that } \forall x, y \in S, (|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon).$$

Thus, we say that f is **uniformly continuous** if f is uniformly continuous on $\operatorname{dom}(f)$.

The difference between continuous and uniform continuous function is that for continuity at a given point x_0, your choice of δ can depend on x_0 (since it is given from the start); while for uniform continuity, since the second point/point of reference appears after we demand for a choice of δ, our δ must work for all "x_0" and thus not depend on it.
(Formally in Mathematics, we say that the δ is **uniform** (independent) in x_0.)

Similar to continuous functions, here are some theorems that one can use to prove uniform continuous without the $\varepsilon - \delta$ definitions:

Theorem 39.

- (Theorem 19.2.) If f is continuous on a closed interval $[a, b]$, then it is uniformly continuous on $[a, b]$.
- (Theorem 19.5, Continuous Extension Theorem.) A real-valued function f on (a, b) is uniformly continuous on (a, b) if and only if it can be extended to a continuous function $\tilde{f}$ on $[a, b]$.[a]
- (Theorem 19.6.) Let f be a continuous function on an interval I (possibly unbounded). Let I° be the interval obtained by removing from I any endpoints that happen to be in I. If f is differentiable on I° and if f' is bounded on I°, then f is uniformly continuous on I.

[a] The same can be said for a function defined on $(a, b]$ or $[b, a)$, since the proof of such a theorem follows from the proof of this original theorem but with less steps.

Personal Opinion: Theorem 19.6 is one of the most useful theorems for proving uniform continuity, since this reduces to a Calculus (31A/B) problem! Differentiate and find its extrema and you will be done! :)

Last but not least, we shall define the limit of a function.

Definition 40. Let S be a subset of $\mathbb{R}$ and let a be a real number of symbol ∞ or $-\infty$ that is the limit of some sequence in S, and let L be a real number or symbol $+\infty$ or $-\infty$. We write $\lim_{x \to a^S} f(x) = L$ if and only if either of the following definitions hold:

- (Definition 20.1.) f is a function defined on S and for every sequence (x_n) in S with limit a, we have $\lim_{n\to\infty} f(x_n) = L$.
- (Theorem 20.6.)

$$\forall\, \varepsilon > 0, \exists\, \delta > 0, \forall x \in S, (|x - a| < \delta \implies |f(x) - L| < \varepsilon).$$

(Theorem 20.10.) Furthermore, let f be a function defined on $J \setminus \{a\}$ for some open interval J containing a, Then, $\lim_{x\to a} f(x)$ exists if and only if the limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist and are equal (in which all three limits are equal). The one-sided limit are defined analogous to Theorem 20.6, as in Corollary 20.8.

Here, the main difference with our usual definition of limit is that for now, since our domain is a subset of $\mathbb{R}$ (rather than $\mathbb{N}$ and thus discrete), it does not make sense to look at "a sequence" (something countable/enumerable) but rather, all possible sequences in the set S converging to a (to illustrate the "uncountability of $\mathbb{R}$".)

The following are some examples on these definitions and how we can prove an explicit function is continuous/uniformly continuous/discontinuous etc.

Example 41. Let f be a function defined on $\mathbb{R}$ as follows:

$$f(x) = \begin{cases} 5x+3 & x \in \mathbb{Q} \\ x+7 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Prove that f is continuous at exactly one point in $\mathbb{R}$.

Proof:

1. Intuitively, the two lines $y = 5x + 3$ and $y = x + 7$ only intersects at one point, which can be obtained by solving $5x + 3 = x + 7$ and thus $x = 1$. Thus, we could expect that the only point of continuity is $x = 1$.

2. **Claim 1:** f is continuous at $x = 1$. Appealing to the $\varepsilon - \delta$ definition of continuity, we first let $\varepsilon > 0$. We shall now pick $\delta > 0$ such that if $|x-1| < \delta$, then $|f(x) - f(1)| = |f(x) - 8| < \varepsilon$. (Here, $f(1) = 5(1) + 3 = 8$ since 1 is rational.)

3. If $|x - 1| < \delta$ and $x \in \mathbb{Q}$, one observe that

$$\begin{aligned} |f(x) - 8| &= |5x + 3 - 8| \\ &= |5x - 5| \\ &= 5|x - 1| \\ &< 5\delta. \end{aligned}$$

   For this to be less than ε, we have to pick $\delta \leq \frac{\varepsilon}{5}$, so $5\delta < \varepsilon$.

4. If $|x - 1| < \delta$ and $x \in \mathbb{R} \setminus \mathbb{Q}$, one observe that

$$\begin{aligned} |f(x) - 8| &= |x + 7 - 8| \\ &= |x - 1| \\ &< \delta. \end{aligned}$$

   For this to be less than ε, we have to pick $\delta \leq \varepsilon$.

5. Now, pick $\delta = \frac{\varepsilon}{5}$. Thus, for all $|x - 1| < \delta = \frac{\varepsilon}{5}$, it is also true that $|x - 1| < \varepsilon$ (since $\frac{\varepsilon}{5} < \varepsilon$).
   Thus, 3. and 4. holds, and we have that for all $x \in \mathbb{R}$ and $|x - 1| < \delta$, $|f(x) - 8| < \varepsilon$.

6. **Claim 2:** f is discontinuous at any $x_0 \neq 1$. Negating the equivalent $\varepsilon - \delta$ definition, this implies that we have to prove:

$$\forall\, x_0 \neq 1, \exists\, \varepsilon > 0, \forall\, \delta > 0, \exists\, x, (|x - x_0| < \delta \text{ AND } |f(x) - f(x_0)| \geq \varepsilon).$$

   The intuition is that as long as x_0 is away from the point of intersection, it must lie on either line (depending on if it is rational or not), and these lines are of some finite non-zero distance away at this value of x_0. Though plausible, this looks scary. We shall instead utilize Definition 17.1. The negation of that is that

$$\exists\, (x_n), \quad x_n \to x_0 \text{ AND } \lim_{n\to\infty} f(x_n) \neq f(x_0).$$

7. If $x_0 \neq 1 \in \mathbb{Q}$, we pick the sequence to be a sequence of irrational numbers converging to x_0. Then, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}(x_n + 7) = x_0 + 7 \neq 5x_0 + 3 = f(x_0)$ (since $x_0 \neq 1$, then $x_0 + 7 \neq 5x_0 + 3$).

8. On the other hand, if $x_0 \neq 1 \in \mathbb{R}\setminus\mathbb{Q}$, we pick the sequence to be a sequence of rational numbers converging to x_0. Then, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}(5x_n + 3) = 5x_0 + 3 \neq x_0 + 7 = f(x_0)$. This thus concludes the proof.

Example 42. Let f be a function defined on $\mathbb{R}$ as follows:

$$f(x) = \frac{x^2 \cos\left(\frac{\pi}{\sqrt{x}}\right)}{(x+1)^3} \quad \text{for } x \in (0,1].$$

Prove that f is uniformly continuous on $(0,1]$.

Proof:

1. First, we observe that $\lim_{x\to 0^+} f(x)$ exists and is equals to 0. This can be proven using the Squeeze Lemma as follows:
$$|f(x)| = \left|\frac{x^2 \cos\left(\frac{\pi}{\sqrt{x}}\right)}{(x+1)^3}\right| \le \left|\frac{x^2}{x^3+1}\right|.$$
Now, by Definition 20.1 (and its variants), $\lim_{x\to 0^+} f(x) = 0$ if for each sequence $x_n \to 0$, we have $f(x_n) \to 0$. Indeed, for each of such sequences, we have
$$|f(x_n)| \le \left|\frac{x_n^2}{x_n^3+1}\right| \to 0$$
as $x_n \to 0$. Thus, we conclude by Squeeze Lemma that $f(x_n) \to 0$.

2. Now, we define the extension as
$$\tilde{f}(x) = \begin{cases} \frac{x^2 \cos\left(\frac{\pi}{\sqrt{x}}\right)}{(x+1)^3} & \text{for } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases}$$
This function $\tilde{f}$ is continuous on $[0,1]$ (continuity at $x = 0$ follows from 1., and continuity at $x \in (0,1]$ follows from the fact that x^2, x^3, $x+1$, $\frac{\pi}{\sqrt{x}}$, and $\cos(x)$ are continuous functions, and the composition/product/sum of continuous functions is continuous (Theorem 17.3 and 17.4)).

3. By Theorem 19.5, we deduce that f is uniformly continuous on $(0,1]$.

Example 43. Let $f(x) = \frac{1}{x}$. Without using the $\varepsilon - \delta$ definition explicitly, prove that f is uniformly continuous on $\left[\frac{1}{2}, \infty\right)$.

Proof:

1. Observe that $f(x) = \frac{1}{x}$ is continuous on $\left[\frac{1}{2}, \infty\right)$. (It is one of the "basic" functions that we assume that is continuous without proof.) Furthermore, f is differentiable on $\left(\frac{1}{2}, \infty\right)$.
2. Compute $f'(x) = -\frac{1}{x^2}$ on $\left(\frac{1}{2}, \infty\right)$.
3. Observe that for all $x \in \left(\frac{1}{2}, \infty\right)$, $|f'(x)| = \frac{1}{x^2} \leq \frac{1}{\left(\frac{1}{2}\right)^2} = 4$. Thus, f' is bounded on $\left(\frac{1}{2}, \infty\right)$.
4. By Theorem 19.6, f is uniformly continuous on $\left[\frac{1}{2}, \infty\right)$.

Functions, Continuity, and Uniform Continuity - Proof-Based Approach.

Now, we shall look at some of the "soft" properties associated with continuous and uniformly continuous functions. They are listed as follows:

- A real-valued function f is said to be **bounded** if $\{f(x) : x \in \mathrm{dom}(f)\}$ is a bounded set; there exists a real number M such that $|f(x)| \leq M$ for all $x \in \mathrm{dom}(f)$.
- (Theorem 18.1, Extreme Value Theorem.) Let f be a continuous real-valued function on a closed interval $[a, b]$. Then, f is a bounded function, and assumes its maximum and minimum values on $[a, b]$. In other words, there exists x_0, y_0 in $[a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$.
- (Theorem 18.2, Intermediate Value Theorem.) Let f be a continuous real-valued function on an interval I (not necessarily closed). Then, f has the intermediate value property on I - for any $a, b \in I$ with $a < b$, and y that lies between $f(a)$ and $f(b)$ (that is, either $f(a) < y < f(b)$ or $f(b) < y < f(a)$), then there exists at least one x in (a, b) such that $f(x) = y$.

Here are some more properties which are somewhat useful, but not as useful as the two "Value Theorems" above:

- (Corollary 18.3.) Let f be a continuous real-valued function on an interval I (not necessarily closed). Then, the set $f(I) = \{f(x) : x \in I\}$ is also an interval or a single point.
- (Theorem 18.5; Partial Converse to Intermediate Value Theorem.) Let g be a strictly increasing function on an interval J such that $g(J)$ is an interval. Then g is continuous on J.
(In other words, a strictly increasing function with the intermediate value property is continuous.)
- (Theorem 18.6.) Let f be a one-to-one continuous function on an interval I. Then f is strictly increasing ($x1 < x_2$ implies $f(x_1) < f(x_2)$) or strictly decreasing ($x_1 < x_2$ implies $f(x_1) > f(x_2)$).
- If f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $f(s_n)$ is a Cauchy sequence.

We shall look at examples on the application of some of these theorems below.

Example 44. Prove that the equation

$$x^2 - 3 = 0$$

has at least two roots.

Proof:

1. Define the function $f(x) = x^2 - 3$.
2. Observe that $f(-2) = 1, f(0) = -3$, and $f(2) = 1$. Furthermore, f is a continuous function on $\mathbb{R}$ (as a polynomial; Exercise 17.5).
3. By Intermediate Value Theorem, there exists $c_1 \in (-2, 0)$ and $c_2 \in (0, 2)$ such that $f(c_1) = 0$ and $f(c_2) = 0$.
4. c_1 and c_2 are thus two roots to the equation $x^2 - 3 = 0$ since $f(c_1) = 0$ implies that $c_1^2 - 3 = 0$ (and similarly for c_2).

Example 45. Let $f : [0,1] \to \mathbb{R}$ be a continuous function on $[0,1]$ and satisfy the following bound: $0 \leq f(x) \leq 3$ for all $x \in [0,1]$. Prove that there exists an $a \in [0,1]$ such that $f(a) = 3a^2$.

Proof:

1. Define the function $g(x) = f(x) - 3x^2$. Observe that g is continuous since f is and so is $3x^2$.
2. Next, observe that $g(0) = f(0) - 3 \cdot (0)^2$ and $g(1) = f(1) - 3 \cdot (1)^2$.
3. If $g(0) = 0$ or $g(1) = 0$, then we are done (the required existence of a is thus either 0 or 1 respectively). Thus, we suppose that $g(0) \neq 0$ and $g(1) \neq 0$.
4. Using the given bound, we observe that $g(0) = f(0) \geq 0$. In fact, since $g(0) \neq 0$, then $f(0) \neq 0$ and therefore $g(0) > 0$.
   On the other hand, we observe that $g(1) = f(1) - 3 \leq 3 - 3 = 0$. In fact, since $g(1) \neq 0$, we must have $g(1) < 0$.
5. By Intermediate Value Theorem, there exists $c \in (0,1)$ such that $g(c) = 0$; ie, $f(c) = 3c^2$.
6. All in all, by considering the case in 3. and the general case (4. and 5.), there exists $a \in [0,1]$ such that $f(a) = 3a^2$.

Example 46. Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous on a the closed bounded interval $[a, b]$ (where $a, b \in \mathbb{R}$), and

$$f(x) > 0 \quad \forall x \in [a, b].$$

Show that there exists a real number $\alpha > 0$ such that

$$f(x) \geq \alpha \quad \forall x \in [a, b].$$

Proof:

1. By Extreme Value Theorem, there exists $x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$.
2. By assumption, we have that $0 < f(x_1) \leq f(x) \leq f(x_2)$ (since $x_1 \in [a, b]$).
3. Hence, pick $\alpha = f(x_1)$. Then, we have $f(x) \geq \alpha = f(x_1)$ for all $x \in [a, b]$.

Example 47. Let S be a non-empty subset of $\mathbb{R}$ and let f be a real-valued function defined on S satisfying the **Lipschitz condition** on S: There exists a constant $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y| \quad \forall\, x, y \in S.$$

Prove that f is uniformly continuous on S.

Proof:

1. We shall prove this by definition. Let $\varepsilon > 0$ be given. We would like to pick an appropriate $\delta > 0$ such that for all $x, y \in S$, we have $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

2. Observe that we can utilize the Lipschitz condition as follows:

$$|f(x) - f(y)| \leq K|x - y| < K\delta.$$

   Thus, we would just have to pick δ such that $K\delta \leq \varepsilon$, or $\delta \leq \frac{\varepsilon}{K}$.

3. Thus, pick $\delta = \frac{\varepsilon}{K}$. By virtue of 2., we have

$$|f(x) - f(y)| \leq K|x - y| < K\delta = K\left(\frac{\varepsilon}{K}\right) = \varepsilon.$$

Example 48. Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions that are continuous on $\mathbb{R}$. Suppose that

$$f(r) = g(r) \quad \forall\, r \in \mathbb{Q}.$$

Prove that

$$f(x) = g(x) \quad \forall\, x \in \mathbb{R}.$$

Proof:

1. Define $h(x) := f(x) - g(x)$ for all $x \in \mathbb{R}$. Since f and g are continuous, then so is $h = f - g$.

2. Fix any $x \in \mathbb{R} \setminus \mathbb{Q}$. (Since if $x \in \mathbb{Q}$, the claim follows immediately from the assumption of the question.) Take a rational sequence r_n converging to x (such a sequence exists purely by construction; write x in decimal form, and pick a sequence of rational numbers such that it agrees with x up to the n-th decimal place).

3. Hence, by continuity of $f, g,$ and h (in the sense of Definition 17.1), we have $h(x) = \lim_{n\to\infty} h(r_n) = \lim_{n\to\infty}(f(r_n) - g(r_n)) = \lim_{n\to\infty}(0) = 0$.

4. Since this holds for arbitrary $x \in \mathbb{R} \setminus \mathbb{Q}$, then we have $f(x) = g(x)$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$ and we are done.

6 Discussion 6

Differentiation, Mean Value Theorems, and Taylor Series. We first start off with the definition of derivatives:

Definition 49. Let f be a real-valued function defined on an open interval containing a point a. We say f is differentiable at a, or f has a derivative at a if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite. We then write $f'(a)$ for the derivative of f at a, and this is thus given by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Key Theorems:

- (Theorem 28.2.) If f is differentiable at a point a, then it is continuous at the point a.
- (Theorems 28.4 , Differentiation Rules.) Let f and g be functions that are differentiable at the point a. Then, the functions (with c, a real constant) $cf, f + g, fg$, and f/g are also differentiable at a, except f/g if $g(a) = 0$ (since it is not defined in this case). The formulas are given by
  - (i) $(cf)'(a) = c \cdot f'(a)$.
  - (ii) $(f + g)'(a) = f'(a) + g'(a)$.
  - (iii) (Product Rule.) $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$.
  - (iv) (Quotient Rule.) $(f/g)'(a) = (g(a)f'(a) - f(a)g'(a))/g(a)^2$ if $g(a) \neq 0$.
- (Theorem 28.5, Chain Rule.) If f is differentiable at a and g is differentiable at $f(a)$, then the composite function $g \circ f$ is differentiable at a and we have $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.
- (Theorem 29.2, Rolle's Theorem.) Let f be a continuous function on $[a, b]$ and differentiable on (a, b) and satisfies $f(a) = f(b)$. Then, there exists $c \in (a, b)$ such that $f'(c) = 0$.
- (Theorem 29.3, Mean Value Theorem.) Let f be a continuous function on $[a, b]$ and differentiable on (a, b). Then, there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
  (Note that this implies Rolle's Theorem if $f(b) = f(a)$.)
  (It is often easier to view this as $f(b) - f(a) = f'(c)(b - a)$; for a graph $y = f(x)$, this is equivalent to "change in y" $\approx \frac{\mathrm{d}y}{\mathrm{d}x} \times$ "change in x".)
- (Theorem 29.8, Intermediate Value Theorem for Derivatives.) Let f be a differentiable function on (a, b). If $a < x_1 < x_2 < b$, and if c lies between $f'(x_1)$ and $f'(x_2)$, then there exists $x \in (x_1, x_2)$ such that $f'(x) = c$.
- (Theorem 29.9, Inverse Function Theorem.) Let f be a one-to-one continuous function on an open interval I, and let $J = f(I)$. If f is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0) \in J$ and
  $$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$
- (Theorem 31.5, Taylor's Theorem.) Let f be defined on (a, b) where $a < c < b$ (here, we allow $a = -\infty$ or $b = \infty$), and suppose that the $n-$th derivative exists and is continuous on (a, b). Then for $x \neq c \in (a, b)$, there exists some y between c and x such that
  $$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x - c)^k + \frac{f^{(n)}(y)}{n!}(x - c)^n.$$
  (Here, c is colloquially known as the "point of expansion" for the Taylor's Series.)

Some Corollaries and Remarks includes:

- (Corollary 29.7.) Let f be a differentiable function on the interval (a, b). Then,
  (i) f is strictly increasing[20] if $f'(x) > 0$ for all $x \in (a, b)$.
  (ii) f is strictly decreasing if $f'(x) < 0$ for all $x \in (a, b)$.
  (iii) f is increasing if $f'(x) \geq 0$ for all $x \in (a, b)$.
  (iv) f is decreasing if $f'(x) \leq 0$ for all $x \in (a, b)$.

- "Differential Equations Corollaries".
  (i) (Corollary 29.4.) Let f be a differentiable function on (a, b) such that $f'(x) = 0$ for all $x \in (a, b)$. Then f is a constant function on (a, b).
  (Basically, if $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, then $y =$ constant.)
  (ii) (Corollary 29.5.) Let f and g be differentiable functions on (a, b) such that $f' = g'$ for all $x \in (a, b)$. Then there exists a constant c such that $f(x) = g(x) + c$ for all $x \in (a, b)$.
  (Basically, if $\frac{\mathrm{d}y_1}{\mathrm{d}x} = \frac{\mathrm{d}y_2}{\mathrm{d}x}$, then $\frac{\mathrm{d}}{\mathrm{d}x}(y_1 - y_2) = 0$ and hence $y_1 - y_2 =$ constant.)

- Intuitive Interpretation of Inverse Function Theorem. Let $y = f(x)$. Then, $\frac{\mathrm{d}y}{\mathrm{d}x}|_x = f'(x)$. The corresponding inverse function is $x = f^{-1}(y)$, so $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\mathrm{d}f^{-1}(y)}{\mathrm{d}y} = (f^{-1})'(y)$. Using the "fact" that

$$\frac{\mathrm{d}x}{\mathrm{d}y}|_y = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}|_x,$$

  we then have

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)},$$

  where $y_0 = f(x_0)$ (so that they are evaluated at the same point).

- The full Taylor's Series (Theorem 31.1) about a given point c is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k,$$

  assuming that f is a function defined on some open interval containing c with all its derivatives at c defined.

[20] We say that a function f is strictly **increasing** if for each $x_1, x_2 \in \mathrm{dom}(f)$ such that $x_1 < x_2$, we have $f(x_1) < f(x_2)$. A similar definition exists for "strictly **decreasing**".

Example 50. Let $0 < a < b$ and $n \in \mathbb{N}$. Use the Mean Value Theorem to establish

$$na^{n-1}(b-a) \leq b^n - a^n \leq nb^{n-1}(b-a).$$

Proof:

1. Define the function $f : [0, \infty) \to \mathbb{R}$ as $f(x) = x^n$. The function is differentiable on $(0, \infty)$ and continuous on $[0, \infty)$.

2. From the given $0 < a < b$, the function is thus differentiable on (a, b) and continuous on $[a, b]$ by 1. Hence, by Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b-a).$$

3. Compute $f'(x) = nx^{n-1}$ for all $x \in (0, \infty)$ and $n \in \mathbb{N}$. Substitute this into the equation in 2. to obtain

$$f(b) - f(a) = nc^{n-1}(b-a).$$

4. Since $a < c < b$, then we have $a^{n-1} \leq c^{n-1} \leq b^{n-1}$. Thus, we have

$$na^{n-1}(b-a) \leq nc^{n-1}(b-a) \leq nb^{n-1}(b-a)$$

since $b - a > 0$ and $n > 0$, and thus

$$na^{n-1}(b-a) \leq f(b) - f(c) \leq nb^{n-1}(b-a)$$

as required.

Example 51. Let $f(x) = x^5 + 4x + 3$ for all $x \in \mathbb{R}$, and let $g = f^{-1} : \mathbb{R} \to \mathbb{R}$ be the inverse function of f. Compute $g'(8)$, justifying your answer.

Proof:

1. This is a question that shouts "use Inverse Function Theroem!!!". To do so, we are required to check the requirements of the Inverse Function Theorem.

2. f as a polynomial is continuous (Exercise 18.8) on $\mathbb{R}$. Furthermore, f is also differentiable for at any $x \in \mathbb{R}$ (as a polynomial, as a result of the relevant differentiation rules in Theorem 28.4 and that the monomial x^n is differentiable with $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$ as seen in Example 28.3).

3. When $y_0 = 8$, by divine intervention, we observe that $x_0 = 1$ is the required value since $f(x_0) = f(1) = 1^5 + 4 \cdot 1 + 3 = 8 = y_0$. Thus, we consider an open interval I containing 1 and the corresponding open interval J containing 8 such that $f(I) = J$.

4. Compute $f'(x) = 5x^4 + 4 > 0$ for all $x \in \mathbb{R}$. By Corollary 29.7, we deduce that f is strictly increasing on $\mathbb{R}$ and hence $I \subseteq \mathbb{R}$.
   Consequently, if f is strictly increasing on some open interval I containing 1 and $f(I) = J$, then $f : I \to J$ is one-to-one.[21]

5. This implies that we can now use the Inverse Function theorem. Observe that this gives
   $$g'(y_0) = \frac{1}{f'(x_0)}$$
   for each $y_0 = f(x_0) \in J$ for some $x_0 \in I$.

6. Since $f'(x) = 5x^4 + 4$, then we have $f'(1) = 9$. From 4., this implies that
   $$g'(8) = \frac{1}{f'(1)} = \frac{1}{9},$$
   as required.

[21]This relies on the following Lemma: If $f : X \to Y$ is strictly increasing and $f(X) = Y$ (surjective), then f is necessarily one-to-one. Recall that $f : X \to Y$ with $f(X) = Y$ is one-to-one (bijective in the textbook) if for any $y \in Y$, there exists a unique $x \in X$ such that $f(x) = y$. Suppose $f : X \to Y$ is strictly increasing and $f(X) = Y$ yet f is not one-to-one. This implies that there exists $y_0 \in Y$ such that there are two different $x_1 < x_2 \in X$ ($x_1 < x_2$ is assumed without loss of generality) such that $f(x_1) = f(x_2) = y_0$. However, since f is strictly increasing, we must have $f(x_1) < f(x_2)$, contradicting $f(x_1) = f(x_2)$.
Note that if $f(X) \neq Y$, this implies there are points in Y that are not mapped by f from any points in X, so the "existence" of an x such that $f(x) = y$ might not hold. Nonetheless, for Inverse Function Theorem questions, we are free to pick J such that $f(I) = J$ (I refers to the open interval containing x_0 we would like to apply the theorem on, and thus the open interval J should include $y_0 = f(x_0)$. We then record the output $f(x)$ for each $x \in I$ and denote the set of these outputs as J. This is how we construct the corresponding J such that $f(I) = J$.)

Example 52. Let $x \in \mathbb{R}$ be arbitrary. Prove the following inequality using Taylor's Theorem:

$$1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 \leq \sqrt{1+x^2} \leq 1 + \frac{1}{2}x^2.$$

1. First, consider the substitution $y = x^2$. Consequently, the claim of the question is equivalent to proving that for all $y \geq 0$, we have
$$1 + \frac{1}{2}y - \frac{1}{8}y^2 \leq \sqrt{1+y} \leq 1 + \frac{1}{2}y.$$
The inequality is true for $y = 0$, so it remains to show this for $y > 0$.

2. Define the function $f(y) = \sqrt{1+y}$, and observe that
   - $f'(y) = \frac{1}{2\sqrt{1+y}}$
   - $f''(y) = -\frac{1}{4(1+y)^{\frac{3}{2}}}$
   - $f'''(y) = \frac{3}{8(1+y)^{\frac{5}{2}}}$

3. By Taylor's Theorem up to the second order and expanding about 0, for each $y > 0$, there exists $\xi \in (0, y)$ such that
$$f(y) = f(0) + f'(0)y + \frac{f''(\xi)}{2}y^2.$$
Since $f(0) = 1$, $f'(0) = \frac{1}{2}$, and $f''(\xi) = -\frac{1}{4(1+\xi)^{\frac{3}{2}}}$, we have
$$f(y) = 1 + \frac{1}{2}y - \frac{1}{8(1+\xi)^{\frac{3}{2}}}y^2 \leq 1 + \frac{1}{2}y$$
since $-\frac{1}{8(1+\xi)^{\frac{3}{2}}}y^2 \leq 0$.

4. On the other hand, by Taylor's Theorem up to the third order and expanding about 0, for each $y > 0$, there exists $\eta \in (0, y)$ such that
$$f(y) = f(0) + f'(0)y + \frac{f''(0)}{2}y^2 + \frac{f'''(\eta)}{6}y^3.$$
Since $f(0) = 1$, $f'(0) = \frac{1}{2}$, $f''(0) = -\frac{1}{4}$, and $f'''(\eta) = \frac{3}{8(1+\eta)^{\frac{5}{2}}}$, we have
$$f(y) = 1 + \frac{1}{2}y - \frac{\frac{1}{4}}{2}y^2 + \frac{3}{6 \cdot 8(1+\eta)^{\frac{5}{2}}}y^3 \geq 1 + \frac{1}{2}y - \frac{1}{8}y^2$$
since $\frac{3}{6 \cdot 8(1+\eta)^{\frac{5}{2}}}y^3 \geq 0$.

5. Combining the results obtained from 3. and 4. (with the $y = 0$ case from 1.), we then obtain the required inequality in 1.

Revision for Finals.

Example 53. Let (a_n) be a sequence of nonnegative numbers such that

$$\sum a_n = +\infty.$$

Prove that

$$\sum \frac{a_n}{2a_n+1} = +\infty.$$

Proof:

1. The contrapositive of the statement is if

$$\sum \frac{a_n}{2a_n+1} < +\infty,$$

then

$$\sum a_n < +\infty.$$

This implies that

$$\sum \frac{a_n}{2a_n+1}$$

is finite (that is, can't be $-\infty$, since it can't be negative, as this would contradict that each term in the series is a nonnegative number).

2. Thus, we deduce that $\frac{a_n}{2a_n+1} \to 0$. Since $\frac{a_n}{2a_n+1} = \frac{1}{2} - \frac{1}{2(2a_n+1)} \to 0$, this implies that $\frac{1}{2a_n+1} \to 1$ and hence $a_n \to 0$ necessarily.

3. This implies that there exists an N such that for all $n > N$, we have $|a_n| < 1$. Since $a_n \geq 0$ for all n, we must then have $0 \leq a_n < 1$ for all $n > N$.

4. To prove that $\sum a_n < +\infty$, observe that

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n \\
&= \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} \frac{a_n(2a_n+1)}{2a_n+1} \\
&\leq \sum_{n=1}^{N} a_n + 3 \sum_{n=N+1}^{\infty} \frac{a_n}{2a_n+1} \\
&\leq \sum_{n=1}^{N} a_n + 3 \sum_{n=1}^{\infty} \frac{a_n}{2a_n+1} \\
&< +\infty.
\end{aligned}$$

Here, we have used the fact that since $0 \leq a_n < 1$, then $0 < 1 \leq 2a_n + 1 \leq 3$, and that $\sum_{n=1}^{N} a_n$ up to any fixed N is finite (since it is a finite sum of real numbers). Furthermore, since each term $\frac{a_n}{2a_n+1}$ is non-negative, we can bound the sum $\sum_{n=N+1}^{\infty} \frac{a_n}{2a_n+1}$ by the full sum $\sum_{n=1}^{\infty} \frac{a_n}{2a_n+1}$, which we know by assumption that it is finite! This concludes the proof.

Example 54. Let $a, b \in \mathbb{R}$ be such that $0 < a < b$. Prove that the open interval (a, b) contains an irrational number of the form $\pi \cdot r$, where r is some non-zero rational number. You may freely use the fact that π is an irrational number.

Proof:

1. Consider the interval $\left(\frac{a}{\pi}, \frac{b}{\pi}\right)$. Note that $\frac{a}{\pi}$ and $\frac{b}{\pi}$ are both real numbers.
2. By the denseness of $\mathbb{Q}$ in $\mathbb{R}$, we deduce that there exists a rational number r such that $\frac{a}{\pi} < r < \frac{b}{\pi}$.
3. Equivalently, by multiplying the inequality by π, there exists an irrational number of the form $\pi \cdot r$ where r is a rational number such that $a < \pi \cdot r < b$.

Example 55. Let A and B be two non-empty bounded subsets of $\mathbb{R}$, such that $\sup(A) > 0$ and $\inf(B) > 0$. Let

$$C = \left\{ \frac{a}{b} : a \in A, b \in B \right\}.$$

Prove that

$$\sup C = \frac{\sup A}{\inf B}.$$

1. First, observe that $a \le \sup A$ and $b \ge \inf B$ for all $a \in A$ and $b \in B$. (This follows from the fact that $\sup A$ is an upper bound of the set A and $\inf B$ is a lower bound of the set B.) This thus implies that $\frac{a}{b} \le \frac{\sup A}{\inf B}$ for each $a \in A, b \in B$. Hence, $\frac{\sup A}{\inf B}$ is an upper bound of the set C.

2. To prove that $\frac{\sup A}{\inf B}$ is the **least** upper bound of the set C, suppose that u is an upper bound of the set C. We would then like to show that $\frac{\sup A}{\inf B} \le u$.

3. First, we would like to prove that $u > 0$. Since $\sup(A) > 0$, there exists $a_0 \in A$ such that $a_0 > 0$. Now, take any element in $b_0 \in B$. Then $b_0 \ge \inf B > 0$. This implies that $1/b_0 > 0$. Hence, as an upper bound of the set C, we have $u \ge \frac{a_0}{b_0} > 0$.

4. Now,

$$\begin{aligned}
& u \text{ is an upper bound of } C \\
\implies & u \ge \tfrac{a}{b} \text{ for all } a \in A, b \in B. \\
\implies & ub \ge a \text{ for all } a \in A, b \in B. \\
\implies & ub \text{ is an upper bound of the set } A \text{ (for any } b \in B). \\
\implies & ub \ge \sup A \text{ for all } b \in B. \\
\implies & b \ge \tfrac{\sup A}{u} \text{ for all } b \in B. \\
\implies & \tfrac{\sup A}{u} \text{ is a lower bound of the set } B. \\
\implies & \tfrac{\sup A}{u} \le \inf B. \\
\implies & u \ge \tfrac{\sup A}{\inf B}.
\end{aligned}$$

   This thus concludes the proof.

Example 56. Let f be a function defined on $\mathbb{R}$ such that

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Prove that f is discontinuous everywhere on $\mathbb{R}$.

Proof:

1. Claim: f is discontinuous on $\mathbb{Q}$.
   Fix any $x_0 \in \mathbb{Q}$ and pick an irrational sequence q_n converging to x_0. Then, we have $f(q_n) = 0 \nrightarrow 1 = f(x_0)$. This implies that f is discontinuous on $\mathbb{Q}$.

2. Claim: f is discontinuous on $\mathbb{R} \setminus \mathbb{Q}$.
   Fix any $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ and pick an rational sequence r_n converging to x_0. Then, we have $f(r_n) = 1 \nrightarrow 0 = f(x_0)$. This implies that f is discontinuous on $\mathbb{R} \setminus \mathbb{Q}$.

Example 57. Prove the Contraction Mapping Theorem. This is stated as follows:
If $f : \mathbb{R} \to \mathbb{R}$ satisfies the following inequality:

$$|f(x) - f(y)| \leq C|x - y|$$

for all $x, y \in \mathbb{R}$, and for a given constant $0 < C < 1$, denote the sequence $x_{n+1} = f(x_n)$. Prove that

(a) (x_n) converges,

(b) If $\lim_{n\to\infty} x_n = L$, then $f(L) = L$ (here, we call p the fixed point of the function f if $f(p) = p$; hence, L is a fixed point of the function f), and

(c) The fixed point of the function f is unique.

Proof:

(a) 1. Fix a $k \in \mathbb{N}$. Set $x = x_{k+1}$ and $y = x_k$. Plug this into the given inequality to obtain

$$|x_{k+2} - x_{k+1}| = |f(x_{k+1}) - f(x_k)| \leq C|x_{k+1} - x_k|.$$

2. Since this is true for all k, now fix an $n \in \mathbb{N}$. We can then show by induction that

$$|x_{n+1} - x_n| \leq C^n |x_1 - x_0|.$$

3. One can then modify the proof in Exercise 10.6 of HW 3 to show that the sequence (x_n) is Cauchy as long as $0 < C < 1$. (Details left as an exercise.)

4. Since (x_n) is Cauchy, it then converges.

(b) 1. Once again, let $n \in \mathbb{N}$ be given. Since the sequence (x_n) converges to some $L \in \mathbb{R}$, we now set $x = x_n$ and $y = L$. This yields

$$|x_{n+1} - f(L)| = |f(x_n) - f(L)| \leq C|x_n - L|.$$

2. Taking limits on both sides of the inequality (allowed by Exercise 9.9(c), and since the limits on both sides of the inequality exists), we thus have

$$|L - f(L)| \leq C|L - L| = 0.$$

This follows from the fact that if $x_n \to L$, then (as a sequence converges, any subsequence converges to the same limit) $x_{n+1} \to L$, and relevant limit laws to interchange limits and absolute signs.

3. Since $0 \leq |L - f(L)| \leq 0$, this is only possible if $|L - f(L)| = 0$. In other words, if $f(L) = L$, and we are done.

(c) 1. Suppose that there are two fixed points to f, that is, $f(L_1) = L_1$ and $f(L_2) = L_2$ with $L_1 \neq L_2$. Plug this into the given inequality (with $x = L_1$ and $y = L_2$) to obtain

$$|L_1 - L_2| = |f(L_1) - f(L_2)| \leq C|L_1 - L_2|.$$

Here, we have used the fact that $f(L_1) = L_1$ and $f(L_2) = L_2$ as fixed points of f.

2. Shifting the term to the other side, we then obtain

$$(1 - C)|L_1 - L_2| \leq 0$$

but $(1 - C) > 0$. This implies that the inequality only makes sense if $|L_1 - L_2| = 0$, or equivalently, $L_1 = L_2$. This thus concludes the proof of uniqueness of fixed points.

Example 58. Let f be a uniformly continuous function on an interval I, and that there is a positive number k such that $|f(x)| \geq k$ for all $x \in I$. Prove that the function

$$g(x) = \frac{1}{f(x)}$$

is uniformly continuous on I.

Proof:

1. Fix an $\varepsilon > 0$. We would like to show that there exists $\delta > 0$ such that for all $x, y \in I$, if $|x - y| < \delta$, then we have $|g(x) - g(y)| < \varepsilon$.

2. First, we compute

$$\begin{aligned} |g(x) - g(y)| &= \left| \frac{1}{f(x)} - \frac{1}{f(y)} \right| \\ &= \left| \frac{f(y) - f(x)}{f(x)f(y)} \right| \\ &\leq \frac{1}{k^2} |f(y) - f(x)|. \end{aligned}$$

   Here, we have used the fact that $|f(x)| \geq k$ for any $x \in I$, so $|f(x)f(y)| \geq k^2$ for any $x, y \in I$.

3. Next, we notice that $|f(y) - f(x)|$ can be obtained with the help of uniformly continuity of f. Indeed, with this $\varepsilon > 0$ fixed from 1., we have that $k^2\varepsilon > 0$. Consequently, we use the definition of uniform continuity of f to produce a $\delta' > 0$ such that for each $x, y \in I$ and $|x - y| < \delta'$, we have that $|f(x) - f(y)| < k^2\varepsilon$.

4. Now, set $\delta = \delta'$. Thus, for all $x, y \in I$ such that $|x - y| < \delta$, we then have that $|x - y| < \delta'$ and 3. is thus activated. This implies that we have the bound $|f(x) - f(y)| < k^2\varepsilon$ for all $x, y \in I$ such that $|x - y| < \delta$.

5. Combining 4. with 2., we then observe that

$$\begin{aligned} |g(x) - g(y)| &\leq \frac{1}{k^2} |f(y) - f(x)| \\ &< \frac{1}{k^2} (k^2\varepsilon) = \varepsilon \end{aligned}$$

   as required.

Example 59. Let $C_0, C_1, \cdots, C_n$ be real constants (with a fixed $n \in \mathbb{N}$) that satisfy the equation

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

prove that the equation

$$C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof:

1. Define the function $f(x) = C_0 x + \frac{C_1}{2} x^2 + \cdots + \frac{C_n}{n+1} x^{n+1}$.
2. Observe that $f(0) = 0$. Furthermore, using the assumption in the question, we also have that $f(1) = C_0 + \frac{C_1}{2} + \cdots + \frac{C_n}{n+1} = 0$.
3. By Rolle's theorem, there exists $c \in (0, 1)$ such that $f'(c) = 0$.
4. Compute $f'(x) = C_0 + C_1 x + \cdots + C_n x^n$.
5. Then, the existence of a $c \in (0, 1)$ such that $f'(c) = 0$ is precisely the root of the equation $C_0 + C_1 x + \cdots + C_n x^n = 0$.

Example 60. Fix some $\delta > 0$ and $a < b$ in $\mathbb{R}$. Let $f : [a-\delta, b+\delta] \to \mathbb{R}$ be twice differentiable on $(a-\delta, b+\delta)$ and f'' is continuous on $(a-\delta, b+\delta)$, with $f'(a) = f'(b) = 0$. Prove that there exists $c \in (a,b)$ such that

$$\frac{4}{(a-b)^2}|f(a) - f(b)| \leq |f''(c)|.$$

1. Apply Taylor's Series at $x = a$. This implies that for each $x \in [a, b+\delta]$, there exists a $\xi \in (a, x)$ such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(\xi)}{2}(x-a)^2.$$

Since $f'(a) = 0$, we then have

$$f(x) = f(a) + \frac{f''(\xi)}{2}(x-a)^2.$$

Setting $x = \frac{1}{2}(a+b)$ (which is between a and b), we obtain

$$f\left(\frac{1}{2}(a+b)\right) = f(a) + \frac{f''(\xi)}{8}(b-a)^2.$$

2. Similarly, apply Taylor's Series at $x = b$. This implies that for each $x \in [a-\delta, b]$, there exists a $\eta \in (x, b)$ such that

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(\eta)}{2}(x-b)^2.$$

Since $f'(b) = 0$, we then have

$$f(x) = f(b) + \frac{f''(\eta)}{2}(x-b)^2.$$

Again, since $\frac{1}{2}(a+b)$ is between a and b, we substitute this into the equation above to obtain

$$f\left(\frac{1}{2}(a+b)\right) = f(b) + \frac{f''(\eta)}{8}(b-a)^2.$$

3. Subtract the equation from 1. with that from 2. and observing that the term $f\left(\frac{1}{2}(a+b)\right)$ cancels out, we have

$$0 = f(a) - f(b) + \frac{f''(\xi) - f''(\eta)}{8}(b-a)^2.$$

Rearranging, and taking modulus, we have

$$\frac{4}{(a-b)^2}|f(a) - f(b)| = \frac{1}{2}|f''(\xi) - f''(\eta)| \leq \frac{1}{2}(|f''(\xi)| + |f''(\eta)|)$$

where we have applied Triangle Inequality in the last step.

4. Without loss of generality, suppose that $|f''(\xi)| > |f''(\eta)|$. The above inequality then reduces to

$$\frac{4}{(a-b)^2}|f(a) - f(b)| \leq \frac{1}{2}(|f''(\xi)| + |f''(\eta)|) \leq \frac{1}{2}(|f''(\xi)| + |f''(\xi)|) = |f''(\xi)|$$

and thus ξ is the required c between a and b.
(It would be η if $|f''(\eta)| > |f''(\xi)|$, and either works if their absolute value of their second derivatives are equal. This explains "without loss of generality" here.)

7 Additional Exercises

Exercise 1. Let $f(x) = x + \lfloor 3x \rfloor$ for all $x \in \mathbb{R}$. Here, $\lfloor y \rfloor$ is the floor function that returns the largest integer that is less than or equals to y. (Example: $\lfloor 3.3 \rfloor = 3$, and $\lfloor 2 \rfloor = 2$.) Determine the set of points in which f is continuous on and the set of points in which f is discontinuous on. Prove that f is indeed continuous and discontinuous at the aforementioned points.

Exercise 2. Let $f : (0,1) \to \mathbb{R}$ and $L \in \mathbb{R}$. Prove that $\lim_{x\to 0^+} f(x) = L$ if and only if $\lim_{y\to+\infty} f\left(\frac{1}{y}\right) = L$.

Exercise 3. Let S be a non-empty subset of $\mathbb{R}$ (not necessarily bounded). For any given real number $k \in \mathbb{R}$, we denote the set

$$kS = \{ks : s \in S\}.$$

Prove that $\sup(131S) = 131\sup(S)$.

Exercise 4. Let $s_1 = \frac{4}{3}$ and $s_{n+1} = s_n^2 - 2s_n + 2$. Prove that $\lim_{n\to\infty} s_n$ exists and compute it.

Exercise 5. Prove that the subsequence of a subsequence of a sequence (s_n) is a subsequence of the original sequence (s_n).

Exercise 6. Fix a sequence in $\mathbb{R}$. Prove that for any subsequence of this sequence, if there exists a subsubsequence that converges to a limit L, then the sequence converges to L.

Exercise 7. Use the Mean Value Theorem to prove that

$$\sqrt{1+x} < 1 + \frac{x}{2}$$

for all $x > 0$.

Exercise 8. Let the function $f : [0,6] \to \mathbb{R}$ be continuous on $[0,6]$ and such that $f(0) = f(6)$. Prove that there exists a point $c \in [0,3]$ such that

$$f(c) = f(c+3).$$

Exercise 9. Consider the function $f(x) = \sqrt{x}$ on $[0,\infty)$. You may assume that f is continuous on $[0,\infty)$ without proof.

(a) Prove that f is uniformly continuous on $[1,\infty)$.

(b) Let I_1 and I_2 be two intervals in $\mathbb{R}$ in which f is defined on. Prove that if f is uniformly continuous on I_1 and I_2 indivdiually, then f is uniformly continuous on $I_1 \cup I_2$.

(c) By using (a) and (b) or otherwise, deduce that f is uniformly continous on $[0,\infty)$.
Hint: f is continuous on $[0,2]$.

Exercise 10. The hyperbolic sine function $\sinh(x)$ is defined as

$$\sinh(x) = \frac{e^x - e^{-x}}{2}.$$

Obtain the Taylor series for $\sinh(x)$ centered at 0, and show that the series converges for any given $x \in \mathbb{R}$.

Exercise 11. Suppose that the function f has the following properties:

(i) f is continuous on $[0, 1]$ and differentiable on $(0, 1)$,

(ii) f' is strictly increasing on $(0, 1)$, and

(iii) $f(0) = 0$.

Prove that the function $\frac{f(x)}{x}$ is strictly increasing on $(0, 1)$.

Exercise 12. Determine if the following series converges or diverges. Justify your answer.

(a) $\sum a_n$, where $a_1 = 1$ and $a_n = \frac{1}{2}\left(1 + \frac{1}{2n}\right)^n a_{n-1}$ for $n = 2, 3, ...$

(b) $\frac{1}{3} + \frac{1}{4^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{3^5} + \frac{1}{4^6} + \cdots$

(c) $\sum \frac{2+(-1)^n}{4^n}$.

Exercise 13. Let (s_n) be a sequence in $\mathbb{R}$.

(a) Prove that if the subsequences $(s_{3n}), (s_{3n+1})$ and (s_{3n+2}) all converges to the same limit L, then the sequence (s_n) converges to L.

(b) With the help of (a) or otherwise, prove the following variant of the Alternating Series Test: Let a_n be a decreasing sequence such that $a_n \geq 0$ for each n and $\lim_{n\to\infty} a_n = 0$. Then, the series $\sum \sin\left(\frac{2n\pi}{3}\right) a_n$ converges.

References

[1] Kenneth A. Ross. *Elementary Analysis: The Theory of Calculus*. Springer, 2nd edition, 2013.