MATH134 - Discussion Supplements for Spring 23

Contents are motivated from [1], the lecture notes by Benjamin Harrop-Griffths (for Fall 21), and some references from Rowan Killip (for Spring 23). [1]

Contents

1 Discussion 1

Recap from Math 33B.

Solving First Order ODEs: (Here, $'$ denotes differentiation with respect to x.)

(i) Separable ODEs, ie. those in the form of

$$y'(x) = f(y(x))g(x).$$

Solution: By separation of variables, we have

$$\int \frac{\mathrm{d}y}{f(y)} = \int g(x)\mathrm{d}x + C$$

for some arbitrary constant C.

(ii) Autonomous ODEs, ie. those in the form of

$$y'(x) = f(y(x)).$$

Solution:

$$\int \frac{\mathrm{d}y}{f(y)} = x + C$$

for some arbitrary constant C. Note that this is obtained by setting $g(x) = 1$ in (i) above.

(iii) General First Order ODEs, ie. those in the form of

$$y'(x) = f(x)y(x) + g(x)$$

for some continuous functions $f(x)$ and $g(x)$. This is done via the use of an integrating factor. First, we bring the term with $y(x)$ to the left-hand side to obtain

$$1y'(x) + -f(x)y(x) = g(x). \tag{1}$$

Then, suppose that we want to write the left-hand side in terms of $\frac{\mathrm{d}}{\mathrm{d}x}$ (something). A good guess is that it is the product of two terms, and hence we can retrieve the two terms on the right by applying the product rule. Thus, we assume that it should be of the form $\frac{\mathrm{d}}{\mathrm{d}x}(w(x)y(x))$. Applying the product rule, we have

$$(w(x)y(x))' = w(x)y'(x) + w'(x)y(x). \tag{2}$$

By comparing both equations, we then require that

$$\frac{w'(x)}{w(x)} = -f(x).$$

Hence, by separation of variables, we have

$$w(x) = e^{-\int^x f(s)\mathrm{d}s}.$$

This implies that if we multiply (1) by $w(x)$, we get (2) on the left-hand side. This implies that

$$\begin{aligned} (w(x)y(x))' &= w(x)g(x) \\ \left(e^{-\int^x f(s)\mathrm{d}s}y(x)\right)' &= e^{-\int^x f(s)\mathrm{d}s}g(x). \end{aligned} \tag{3}$$

The above equation can now be solved by integrating both sides directly.

Solving Second Order Linear ODEs.

We will review this by an example. Consider the ODE:

$$y''(t) + 3y'(t) + 2y(t) = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

(Here, $'$ denotes differentiation with respect to t.) We can solve the ODE as follows.

(i) *Ansatz* Method. Assume that $y(t) = e^{\lambda t}$, and substitute this into the equation above to generate the characteristic equation:

$$\lambda^2 + 3\lambda + 2 = 0.$$

Solution to the characteristic equation above: $\lambda = -1$ or $\lambda = -2$. By linearity of solutions, we have

$$\begin{aligned} y(t) &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ &= C_1 e^{-t} + C_2 e^{-2t}. \end{aligned}$$

Substitute $y(0) = 1$ and $y'(0) = 0$ to obtain two linear equations for C_1 and C_2 and solve them to obtain $y(t) = 2e^{-t} + (-1)e^{-2t}$.

(ii) *By converting to a* 2×2 *linear system.* It is important that you know how to use this method and its generalization to an $n \times n$ system as it will be largely used in the later parts of this class.

Let $x_1 = y$ and $x_2 = y'$. We consider the vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Our goal is to convert the system into the form of $\mathbf{x}' = A\mathbf{x}$ where A is a 2×2 matrix. Note that the matrix equation is equivalent to

$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}.$$

The first equation reads $y' = ay + by'$, in which we can deduce that $a = 0$ and $b = 1$. The second equation reads $-3y' - 2y = y'' = cy + dy'$, where the left equality is obtained from the ODE of interest, while the right equality is obtained by expanding the second component of the matrix equation. This implies that $c = -2$ and $d = -3$. Thus, we have

$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}. \tag{4}$$

This equation can be solved by finding the eigenvalues of $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$. If λ is an eigenvalue of A, then we have $\det(\lambda I - A) = 0$ as the corresponding characteristic equation to be solved. One can check that the equation simplifies to

$$\lambda^2 + 3\lambda + 2 = 0.$$

(Does this look familiar? Refer to the *Ansatz* method above.)
The solution to this quadratic equation is given by $\lambda = -1$ or $\lambda = -2$.
Next, we will proceed to solve for the corresponding eigenvectors for the eigenvalues -1 and -2. One can check that the eigenvectors are given by $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ respectively. [2] There are "two" ways to proceed from here. Next, recall that the solution to the system with distinct eigenvalues is given by

$$\begin{aligned} \mathbf{x}(t) &= C_1 e^{\lambda_1 t} \mathbf{v_1} + C_2 e^{\lambda_2 t} \mathbf{v_2} \\ &= C_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}. \end{aligned}$$

Note that this is because if we use the *ansatz* $\mathbf{x}(t) = Ce^{\lambda t}\mathbf{v}$ where $\mathbf{v}$ is the eigenvector of A with eigenvalue λ, it does satisfy $\mathbf{x}(t)' = Ce^{\lambda t}(\lambda \mathbf{v}) = Ce^{\lambda t} A\mathbf{v} = A(Ce^{\lambda t}\mathbf{v}) = A\mathbf{x}(t)$ and thus satisfy the system $\mathbf{x}' = A\mathbf{x}$ above.
Input $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ from the given initial condition and solve the linear system to obtain $C_1 = -2$ and $C_2 = 1$. We then have

$$\mathbf{x}(t) = -2e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{-2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

[2]These can be obtained by substituting the relevant values of λ into $\lambda I - A$ and solving the system $(\lambda I - A)\mathbf{x} = 0$ for $\mathbf{x}$.

in which reading off the matrix equation from the first component yields

$$y(t) = 2e^{-t} + (-1)e^{-2t}.$$

Remark: Please review the 2×2 system method in 33B if these seem foreign to you! In addition, if you are not sure as to how you can solve for the eigenvalues and eigenvectors for general matrices, I would also suggest reviewing materials from 33A.

Example 1. Solve the following initial-value problems:

(i) Given

$$y' = \frac{x}{y} + \frac{1}{y}, \qquad y(0) = 1,$$

determine $y(x)$ for $x \geq 0$.

(ii) Given

$$y' = y - x, \qquad y(0) = 0,$$

determine $y(x)$ for $x \in \mathbb{R}$.

Suggested Solution:

(i) We can combine the two fractions on the right and obtain

$$y'(x) = \frac{x+1}{y}, \qquad y(0) = 1.$$

From here, we will solve this by separation of variables.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x+1}{y} \\
\int_{y(0)}^{y(x)} y\mathrm{d}y &= \int_0^x t + 1\mathrm{d}t \\
\frac{1}{2}y^2|_{x=0}^{x} &= \left(\frac{1}{2}t^2 + t\right)|_{t=0}^{t=x} \\
\frac{1}{2}y^2(x) &= \frac{1}{2}x^2 + x + \frac{1}{2} \\
y^2(x) &= (x+1)^2 \\
y(x) &= (x+1).
\end{aligned}$$

Note that in the last step, we have to take square roots on both sides of the equation. The negative root is rejected in view of the initial condition $y(0) = 1$.

(ii) This is a first-order ODE, and can always be solved using the method of integrating factor. From $y' = y - x$, we have that $y' - y = -x$. Then, we can compute the integrating factor to be $e^{\int^x (-1)\mathrm{d}s} = e^{-x}$. Thus, this implies that

$$\begin{aligned}
(y' - y)e^{-x} &= -xe^{-x} \\
(ye^{-x})' &= -xe^{-x} \\
ye^{-x} - ye^{-x}|_{x=0} &= -\int_0^x te^{-t}\mathrm{d}t.
\end{aligned}$$

Since $y(0) = 0$, we have that $ye^{-x}|_{x=0} = 0 \times e^0 = 0$. On the other hand, the integral on the right can be evaluated using integration by parts as follows:

$$\begin{aligned}
-\int_0^x te^{-t}\mathrm{d}t &= te^{-t}|_{t=0}^{t=x} - \int_0^x e^{-t}\mathrm{d}t \\
&= xe^{-x} + e^{-x} - 1.
\end{aligned}$$

Back to the ODE, this implies that $ye^{-x} = xe^{-x} + e^{-x} - 1$ or

$$y(x) = x + 1 - e^{-x}.$$

Example 2. (2×2 Linear System.) Let $x(t)$ denote the displacement of an object at time t with respect to a fixed reference point. For a damped object with unit mass and unit damping constant, $x(t)$ is governed by the following differential equation:

$$\ddot{x}+\dot{x}=0.$$

(Recall that every dot above x corresponds to taking a derivative with respect to time t once.)

(i) Use the ansatz method to obtain the general solution to the differential equation above.

(ii) Denote $\mathbf{x}=\begin{pmatrix}x\\ \dot{x}\end{pmatrix}$. Note that $\mathbf{x}$ satisfy the matrix equation

$$\dot{\mathbf{x}}=A\mathbf{x}$$

for some 2×2 matrix A. Determine the matrix A.

(iii) Determine the eigenvalues and eigenvectors for A.

(iv) Suppose that the object is moving at unit speed (ie $\dot{x}(0)=1$) at the given reference point at $t=0$. Using your answer in (iii) only, determine an analytic equation for the displacement of the object from the reference point (ie $x(0)=0$) with respect to time, $x(t)$, for $t\in\mathbb{R}$.

Suggested Solution:

(i) Consider the ansatz $x(t)=e^{\lambda t}$. Then, we have $\dot{x}(t)=\lambda e^{\lambda t}$ and $\ddot{x}(t)=\lambda^2e^{\lambda t}$. Plugging this into the ODE, we obtain

$$(\lambda^2+\lambda)e^{\lambda t}=0.$$

Since $e^{\lambda t}\neq0$ for any $t\in\mathbb{R}$, we can divide it on both sides to obtain

$$\lambda^2+\lambda=\lambda(\lambda+1)=0.$$

The solutions are $\lambda=0$ and $\lambda=-1$. This implies that the general solution is given by

$$x(t)=Ae^{0t}+Be^{-t}=A+Be^{-t}$$

for arbitrary constants A and B.

(ii) The matrix equation reads

$$\dot{\mathbf{x}}=\begin{pmatrix}\dot{x}\\ \ddot{x}\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\ \dot{x}\end{pmatrix}.$$

We can see that $a=0$ and $b=1$. Using the fact that $\ddot{x}=-\dot{x}$, we have $c=0$ and $d=-1$. Thus, we have $A=\begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix}$.

(iii) If λ is an eigenvalue of A, it must then satisfy $\det(\lambda I-A)=0$. Equivalently, this is given by

$$\begin{vmatrix}\lambda & -1\\ 0 & \lambda+1\end{vmatrix}=0$$
$$\lambda(\lambda+1)=0.$$

The eigenvalues are 0 and -1. For $\lambda=0$, the system reduces to $\begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix}$ with eigenvector $\begin{pmatrix}1\\ 0\end{pmatrix}$. For $\lambda=-1$, the system reduces to $\begin{pmatrix}-1 & -1\\ 0 & 0\end{pmatrix}$ with eigenvector $\begin{pmatrix}1\\ -1\end{pmatrix}$.

(iv) The general solution for a 2×2 linear system with two distinct eigenvalues is given by

$$\begin{aligned}\mathbf{x}(t)&=C_1e^{\lambda_1t}\mathbf{v_1}+C_2e^{\lambda_2t}\mathbf{v_2}\\ &=C_1\begin{pmatrix}1\\ 0\end{pmatrix}+C_2e^{-t}\begin{pmatrix}1\\ -1\end{pmatrix}.\end{aligned}\tag{5}$$

The physical initial conditions translate to $\dot{x}(0) = 1$ and $x(0) = 0$, or as a vector, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The system at $t = 0$ thus becomes

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 \\ -C_2 \end{pmatrix}$$

and thus $C_2 = -1$ and $C_1 = 1$. Reading off the first component of the matrix equation in Equation (5), we have $x(t) = 1 - e^{-t}$.

Example 3. Given

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 x(x+1) + x^2 + x + y^2 + 1, \qquad y(0) = 1,$$

determine $y(x)$ for x sufficiently close to 0.[a]

[a] This is required for the solution to exist on the entire interval - though you do not have to worry about this now.

Suggested Solution:

The equation can be re-written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x^2 + x + 1)(y^2 + 1), \qquad y(0) = 1.$$

By separation of variables, we have

$$\begin{aligned} \int_{y(0)}^{y(x)} \frac{\mathrm{d}y}{1+y^2} &= \int_0^x t^2 + t + 1 \mathrm{d}t \\ \arctan(y) - \arctan(1) &= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x \\ y(x) &= \tan\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{\pi}{4}\right). \end{aligned}$$

2 Discussion 2

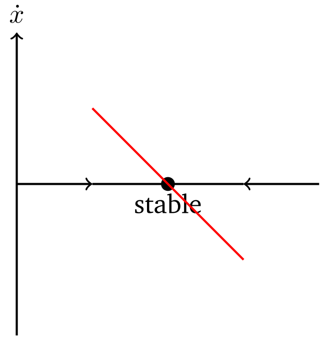
Flows, Stability, and Phase Portrait.

Given a first-order system $\dot{x} = f(x)$, here are some key terms you should know.

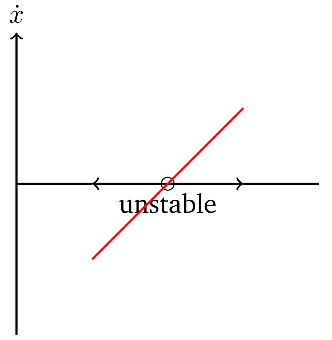
Key Term	Definition/Idea
Flow at a point $x = a$	Sign of $\dot{x}(a)$. The flow is to the right (left) if $\dot{x}(a) > 0$ (< 0).
Fixed Points	x_* where $f(x_*) = 0$
Fixed Point x_* is **Stable**	$\dot{x}(x_*^+) > 0$ and $\dot{x}(x_*^-) < 0$; Flows converge towards the fixed point.
Fixed Point x_* is **Unstable**	$\dot{x}(x_*^+) < 0$ and $\dot{x}(x_*^-) > 0$; Flows diverge from the fixed point.
Fixed Point x_* is **Half-stable**	$\dot{x}(x_*^+)$ and $\dot{x}(x_*^-)$ are of the same sign; Flows diverge in one direction and converge in the other.
Phase Portrait (1D case)	A diagram that indicates flows along the horizontal axis, together with the corresponding fixed points.

The following shows the graph of $\dot{x}$ against x, together with the corresponding phase portrait.

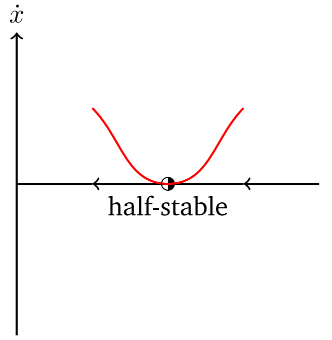
Stable (Represented with a solid circle):



Unstable (Represented with a hollow circle):



Half-Stable (Represented with a half-filled circle, with the filled half corresponds to a stable point from that side):



Hence, by sketching the graph of $f(x)$ against x and computing the relevant direction of the arrows (ie $\dot{x} = f(x) > 0$ implies arrow pointing to the right; < 0 for the left, etc), we can determine the stability of the given fixed point.

Alternatively, we can determine the stability of a given fixed point x_* by computing $f'(x_*)$ as follows:

Sign of $f'(x_*)$	Nature of Fixed Point
< 0	Stable
> 0	Unstable
$= 0$	Indeterminate!

The following shows a brief proof as to why it works. (It is important to know how the proof works.)

Let $\eta = x - x_*$. Then, $\dot{\eta} = \dot{x}$ (since x_* does not depend on time) and by Taylor's Remainder Theorem, we have

$$f(x) = f(x_*) + f'(x_*)(x - x_*) + O((x - x_*)^2) \approx f(x_*) + f'(x_*)(x - x_*)$$

for x close enough to x_*.[3] Since $f(x_*) = 0$ by definition of fixed point, combining all of the above, we have

$$\dot{\eta} = \dot{x} = f(x) \approx f'(x_*)\eta. \tag{6}$$

Locally at $x \approx x_*$, this means that the phase portrait is a straight line that cuts through $x = x_*$ with gradient $f'(x_*)$ (which is why it is literally known as linearization!). Matching with the diagrams for the 3 different types of stability above gives the required nature for stable and unstable fixed points!

The dynamics of how $\eta = x - x_*$ varies with time can be approximated by solving the ODE (6).

Note: If $f'(x_*) = 0$, this implies that we will require higher-order expansions (ie $\dot{\eta} \approx \frac{f''(x_*)}{2}\eta^2$ etc) to investigate the phenomenon better. Alternatively, we can turn potential functions.

Potential Functions.

Given $\dot{x} = f(x)$, let $\dot{x} = -\frac{\mathrm{d}}{\mathrm{d}x}V(x)$ where $V(x)$ is the potential function[4]. Then, we have

$$-\frac{\mathrm{d}}{\mathrm{d}x}V(x) = f(x). \tag{7}$$

Some properties:

- We can obtain the equation for a potential function $V(x)$ by solving the ODE (7) in x.
- Note that upon integrating, there is an arbitrary constant C. Note that it does not matter what value of C you pick here.
- Theorem: $V(x(t))$ is non-increasing with time.
  At points that are not fixed points, $V(x(t))$ is strictly decreasing with time.
- From the above fact, we can sketch the potential function and view it as though an object is falling through the "graph".

Note that a fixed point (corresponds to $f(x_*) = 0$) corresponds to a turning point for $V(x)$ (since by (7), $\frac{\mathrm{d}}{\mathrm{d}x}V(x_*) = -f(x_*) = 0$). Combining with the last bullet point above, we have the following:

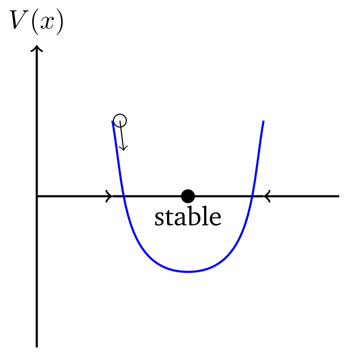
Nature of Isolated Turning Points of $V(x)$	Nature of Fixed Point
Local Minimum	Stable
Local Maximum	Unstable
Inflexion	*Half Stable*[5]

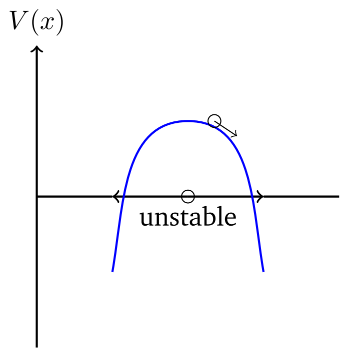
[3]Here, if $y = O(|x|^k)$, it means that $|y| \leq C|x^k|$ for some $C > 0$. As an applied mathematician, this just means $O(|x|^k) \approx C|x|^k$. Thus, if x is close enough to x_*, we have $x - x_*$ is small and thus, $(x - x_*)^2$ is even smaller. This thus "justifies" that we can drop the second order term.

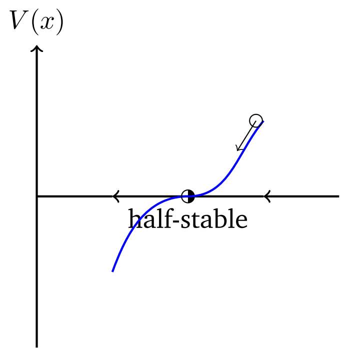
[4]Disambiguation: The potential energy $V(x(t))$ is a function of V with respect to time; The potential function $V(x)$ is a function of V with respect to space x. (Both are just colloquially potentials, represented by V.)

[5]This is different compared to the table above, where $f'(x_*) = 0$ gives "indeterminate". This is because equivalently, $V''(x_*) = 0$ does not necessarily implies that x_* is an inflexion point! Thus, to check if something is an inflexion point, your best bet would be the first derivative test.

Corresponding graphs of potential functions for stable/unstable/half-stable points (use the small little ball along the curve to help you visualize trajectories and the stability of the fixed points!):







<u>Numerical Methods.</u>

Consider

$$\begin{cases} \dot{x}(t) = f(x(t)), \\ x(0) = x_0. \end{cases}$$

We denote the numerical solution at the n-th iteration as x_n. Fix a time step Δt, we have

- Euler's Method: $x_{n+1} = x_n + f(x_n)\Delta t$.
- Heun's Method/Improved Euler's Method: $\widetilde{x}_{n+1} = x_n + f(x_n)\Delta t$ and $x_{n+1} = x_n + \frac{1}{2}[f(x_n) + f(\widetilde{x}_{n+1})](\Delta t)$.

Notation: If the starting time is at $t = 0$, with step size Δt,

- x_n represents the **approximate** value of the solution at $t = n\Delta t$, obtained by our numerical method.
- $x(n\Delta t)$ represents the **exact** value of the solution at time $t = n\Delta t$.

We shall see some examples of the concepts above in the following pages.

Example 4. Consider the following first-order system:

$$\dot{x} = (1 - x^2)(x + 1).$$

(i) Show that $x = 1$ is a stable fixed point.

(ii) Determine any other fixed point(s) (if any) and the corresponding stability of the fixed point(s).

(iii) Draw a phase portrait corresponding to this system.

(iv) Overlay your sketch with the corresponding potential function.

(v) Consider the initial condition $x(0) = 0.99$. By linearizing the given equation and using the fact shown in (i), determine an approximate value of time t such that $x(t) = 0.999$.

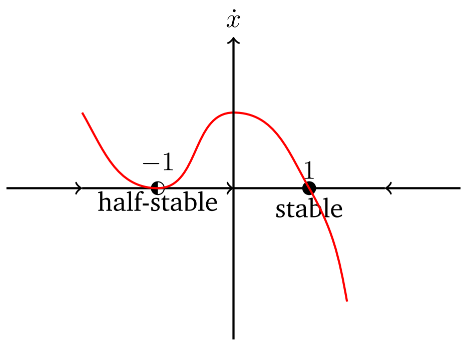
Suggested Solution:

(i), (ii), and (iii). Note that $\dot{x} = f(x) = (1 - x^2)(x + 1) = (1 - x)(1 + x)(x + 1) = -(x + 1)^2(x - 1)$.

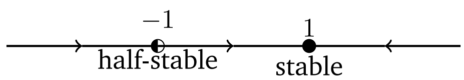
Hence, the roots of this equation are given by $x = \pm 1$. In particular, $x = 1$ is a root of the equation and thus is a fixed point of the system.
Next, we compute $f'(x) = -2(x + 1)(x - 1) - (x + 1)^2$, and see that

- $f'(1) = -4 < 0$, and hence is a stable fixed point.
- $f'(-1) = 0$. We would need to sketch the phase portrait of the system to determine its stability.

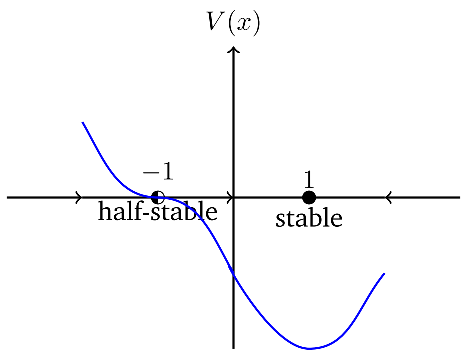
Indeed, the graph of $\dot{x}$ against x is given by



so $x = -1$ is a half-stable point.
Note that if the question asks for phase portrait, so technically speaking, only the following is required.



(iv) The sketch of phase portrait with the potential function overlayed as represented in blue is as follows:



Note that we can sketch the potential function immediately if we already knew the stability of each of the fixed points and the corresponding flow. Alternatively, we can solve $-\frac{\mathrm{d}V}{\mathrm{d}x} = f(x)$ for $V(x)$ and plot the graph accordingly to check that the potential function looks roughly like the one that we have drawn above (up to our choice of the arbitrary constant C, that is, translation in the $y-$ direction).

(v) The linearized equation is given by

$$\dot{\eta} = f'(1)\eta = -4\eta$$

with $\eta(t) = x(t) - x_* = x(t) - 1$, and that $f'(1)$ is computed in (i). Note that the ODE is separable, and hence we can obtain the following solution

$$\eta(t) = \eta(0)e^{-4t}.$$

Set $\eta(0) = x(0) - 1 = 0.99 - 1 = -0.01$ and $\eta(t) = 0.999 - 1 = -0.001$.

Thus, the time required is given by

$$\begin{aligned} -0.001 &= -0.1e^{-4t} \\ 10 &= e^{4t} \\ t &= \boxed{\frac{1}{4}\ln(10)}. \end{aligned}$$

Example 5. Consider the equation

$$\begin{cases} \dot{x} = x^2 \\ x(0) = 0.5. \end{cases}$$

(i) Use Euler's method with step size $\Delta t = 0.2$ to approximate $x(1)$. Tabulate your results, keeping at least six decimal places.

(ii) Find the true value of $x(1)$ by solving the differential equation above and compute the error of your approximation in part (i), giving your answer to six decimal places.

Suggested Solution:
(i) Euler's method gives the following scheme

$$x_{n+1} = x_n + f(x_n)\Delta t$$

with $f(x_n) = x_n^2$, $x_0 = 0.5$ and $\Delta t = 0.2$. The value of x at $t = 1$ corresponds to x_5. The value at each iteration is shown below.

n	x_n
1	0.55
2	0.6105
3	0.685042
4	0.778899
5	$\boxed{0.900235}$

(ii) Note that the equation is separable. Hence, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2$$
$$\int \frac{1}{x^2}\mathrm{d}x = \int \mathrm{d}t$$
$$-\frac{1}{x} = t + C.$$

Using the fact that $x(0) = 0.5$, we have that $C = -2$. Thus, we have

$$x(t) = \frac{1}{2-t}.$$

This implies that $x(1) = 1$. The error of approximation is thus given by

$$|x(1) - x_5| = \boxed{0.099765}.$$

3 Discussion 3

Theorems on Existence and Uniqueness of ODEs and Finite-Time Blowup.

Some terminologies/results from 131A (Analysis) that you might find useful:

- Let $f : [a, b] \to \mathbb{R}$ be a function (with $a < b$). We say that f is **continuous at (a point)** $c \in [a, b]$ if $\lim_{x\to c} f(x) = f(c)$. Rigorously, this means that for all $\varepsilon > 0$, there exists $\delta > 0$ (possibly depending on ε) such that for all $x \in [a, b]$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \varepsilon$.
- We say that a function is continuous on an interval[6] I if it continuous at the point x for all $x \in I$.
- Given an interval $I \subseteq \mathbb{R}$, we say that f is **Lipschitz** (continuous) on I if there exists a constant $L > 0$ such that
$$|f(x) - f(y)| \leq L|x - y|$$
for all $x, y \in I$. The constant L is also known as the **Lipschitz constant**.
- (**Extreme Value Theorem.**) If $f : [a, b] \to \mathbb{R}$ is continuous on the closed interval $[a, b]$, then it is bounded on $[a, b]$. This means that there exists a constant $M > 0$ such that
$$|f(x)| \leq M$$
for all $x \in [a, b]$.
- (**Mean Value Theorem.**) If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a $c \in (a, b)$ such that
$$f(b) - f(a) = f'(c)(b - a).$$
- On the interval $I = [a, b]$ for some $a < b$, we have

  Continuously Differentiable $\implies$ Lipschitz Continuous $\implies$ Continuous.
- (**Triangle Inequality.**) For all $x, y \in \mathbb{R}$, we have
$$||x| - |y|| \leq |x - y| \leq |x| + |y|.$$
- (**Triangle Inequality - Integral Version**) For an integrable function f, we have
$$\left|\int_a^b f(x)\mathrm{d}x\right| \leq \int_a^b |f(x)|\mathrm{d}x.$$

Theorem 6. (Cauchy-Peano Existence Theorem.) Let $f : (a, b) \to \mathbb{R}$ be **continuous** and $x_0 \in (a, b)$. Then, there exists some $T > 0$ and a solution $x : [-T, T] \to \mathbb{R}$ to the ODE[a]:

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0. \end{cases}$$

[a]ODE is an abbreviation for "Ordinary Differential Equation", or just differential equation in our class.

Theorem 7. (Picard's Existence and Uniqueness Theorem.) Let $f : (a, b) \to \mathbb{R}$ be **Lipschitz continuous** and $x_0 \in (a, b)$. Then, there exists some $T > 0$ and a **unique local** solution $x : [-T, T] \to \mathbb{R}$ to the ODE:

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0. \end{cases}$$

[6]Intervals can possibly be open/closed, such as $[a, b]$, or (a, b), or even those of a mixed type like $(a, b]$ etc.

The proof of this is done via Picard's iteration, in which we have

$$x_{n+1} = x_0 + \int_0^t f(x_n(s))\mathrm{d}s$$

and we hope that the iteration converges to $x(t)$, the solution to the given ODE above.

Remark: If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous[7], then there exists a unique **global** solution to

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0. \end{cases}$$

Theorem 8. (Comparison of Solutions.) Let $f \leq g$ be smooth and let $x_0 \leq y_0$. Suppose x and y are the solutions to the ODEs:

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{y} = g(y) \\ y(0) = y_0 \end{cases}$$

on a time interval $[0, T]$. Then, $x(t) \leq y(t)$ for all time $t \in [0, T]$.

One mechanism in which a solution does not exist is that it exhibits finite-time blow-up. In other words, we see that $|x(t)| \to +\infty$ as $t \to T$ for some $T \in \mathbb{R}$ (finite time).

Often, we can use the comparison of solutions to obtain a finite-time blowup argument. We shall see an example below.

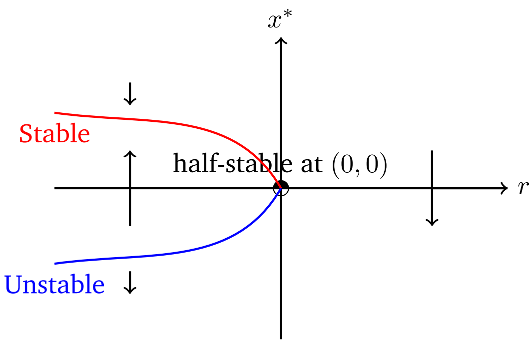
[7] The difference here is that we are looking at f Lipschitz on the entire domain of the function with the same Lipschitz constant.

Bifurcations I: Saddle-Node and Transcritical Bifurcations.
Consider a system
$$\dot{x} = f(x, r)$$
where r refers to a parameter of the system. Suppose that the system has a smooth curve of fixed points $(x, r) = (x^*(r), r)$ for r in some interval. If the stability of the fixed point $x^*(r)$ changes at some $r = r^*$, we say that the system undergoes **bifurcation** at $(x, r) = (x^*, r^*)$.

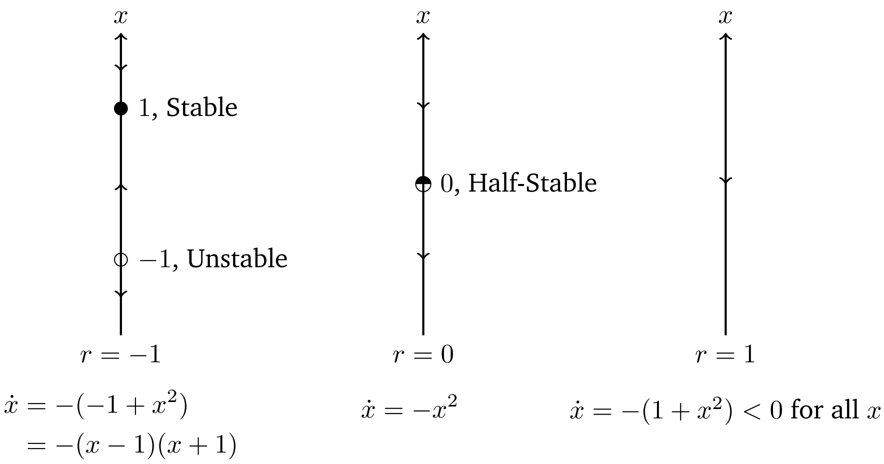
Roughly speaking, at some critical value of the parameter $r = r^*$, we see a change in the characteristics of the fixed point(s) denoted by x^*. These can be visualized on the $x^* - r$ plane, where r is the contentious parameter for bifurcation, in which for each value of r, we draw a phase portrait in x and stack them continuously while emphasizing the fixed points.

For instance, here are two common types of bifurcations (we shall cover one more in the next discussion).

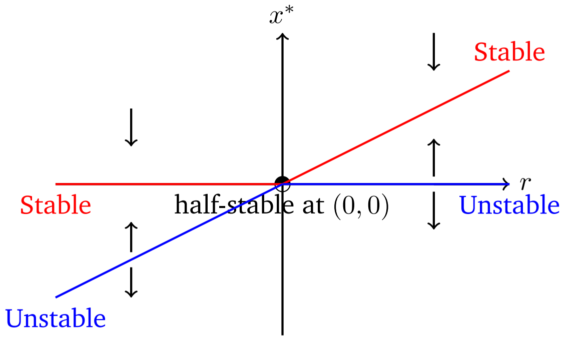
Saddle-Node: (Mutual destruction/creation of fixed points), for instance, $\dot{x} = -(r + x^2)$.
Bifurcation point: $(x^*, r) = (0, 0)$.



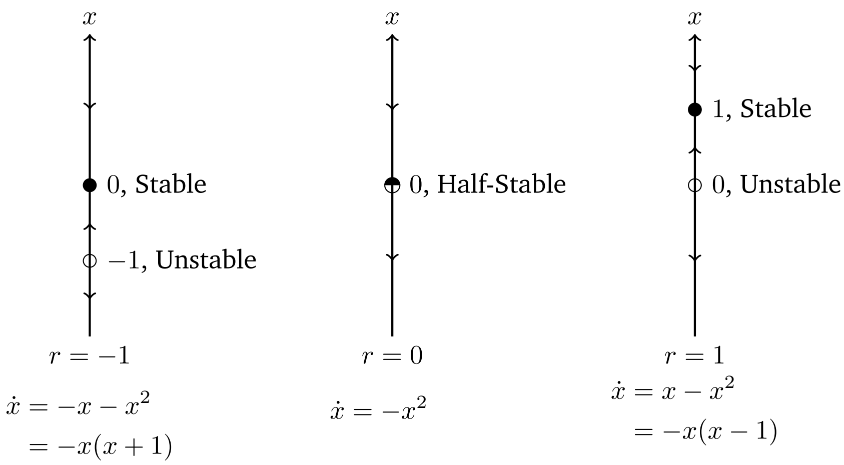
Indeed, to see that the above bifurcation diagram makes sense, we look at three key regimes ($r < 0$, $r = 0$, and $r > 0$) and draw the phase portraits vertically (so that we can observe how it gets stacked).



<u>Transcritical:</u> ("Transfer of critical points"), for instance, $\dot{x} = rx - x^2$.
Bifurcation point: $(x^*, r) = (0, 0)$.



Indeed, to see that the above bifurcation diagram makes sense, we look at three key regimes ($r < 0$, $r = 0$, and $r > 0$) and draw the phase portraits vertically (so that we can observe how it gets stacked).



The following theorem might help with the identification of bifurcation points.

Theorem 9. (Identification of Possible Bifurcation Points.)
If the system

$$\dot{x} = f(x, r)$$

has a bifurcation point at $(x, r) = (x^*, r^*)$, then

$$f(x^*, r^*) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial x} f(x^*, r^*) = 0.$$

Intuitively, the first equation corresponds to solving the fixed point $f(x^*) = 0$. Furthermore, recall that $f'(x^*) < 0$ corresponds to a stable fixed point, while $f'(x^*) > 0$ corresponds to an unstable fixed point. Thus, any change in stability should potentially occur at this transition point where $f'(x^*) = 0$.[8]

Note that the converse might not necessarily hold. Thus, if we solve the above equations $f = 0$ and $\frac{\partial}{\partial x} f = 0$, we have contentious bifurcation points (x^*, r^*). To determine if these are actually bifurcation points, we will have to resort to graphing methods. We shall go through such an example below.

[8]Note that we wrote $f'(x^*)$ above since f is a function of one variable, x. If we introduce the bifurcation parameter r, we now have $\dot{x} = f(x, r)$, where f is a function of two variables. Here, "$f'(x^*)$" now refers to differentiating with respect to x while keeping r fixed, which is precise the definition of the partial derivative $\frac{\partial f}{\partial r}(x, r)$!

Example 10.

(i) Show that

$$\begin{cases} \dot{y} = y^5 \\ y(0) = 1 \end{cases}$$

exhibits finite time blow-up.

(ii) Hence or otherwise, show that

$$\begin{cases} \dot{x} = 1 + x^5 \\ x(0) = 1 \end{cases}$$

exhibits finite time blow-up.

Suggested Solution:

(i) By separation of variables, one can show that with $y(0) = 1$, we have

$$y(t) = \frac{1}{(1-4t)^{\frac{1}{4}}}.$$

Hence, we observe that $y(t) \to \infty$ as $t \to \frac{1}{4}^-$, thus exhibiting finite time blow-up.

(ii) Since $1 + x^5 \geq x^5$ for all $t \geq 0$, by comparison of solutions (Theorem 8), we have that $x(t) \geq y(t)$ for all t such that the solution $y(t)$ exists. Since $y(t) \to \infty$ as $t \to \frac{1}{4}^-$, we have by the above inequality that $x(t) \to \infty$ as $t \to \frac{1}{4}^-$. Hence, $x(t)$ also exhibits finite time blow-up.

Example 11. Consider the following system:

$$\begin{cases} \dot{x} = f(x) \\ x(0) = 1. \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying $|f'(w)| \leq 2e^{-w^2}$ for all $w \in \mathbb{R}$.

(i) Show that f is Lipschitz on $\mathbb{R}$.

(ii) Using your answer in (i), explain why we can deduce that the above system does not exhibit finite-time blowup.

Suggested Solution:

(i) From the given inequality, we see that $|f'(w)| \leq 2e^{-w^2} \leq 2$ since $e^{-w^2} \leq e^0 = 1$ for all $w \in \mathbb{R}$. Thus, take any $x, y \in \mathbb{R}$, we then have by mean value theorem that there exists some ξ between x and y such that

$$f(x) - f(y) = f'(\xi)(x - y).$$

Taking absolute values on both sides, we have

$$|f(x) - f(y)| = |f'(\xi)||(x - y)|.$$

Using the fact that $|f'(w)| \leq 2$ for all w and in particular is true at $w = \xi$, we have $|f'(\xi)| \leq 2$. Thus, we have

$$|f(x) - f(y)| = |f'(\xi)||(x - y)| \leq 2|x - y|.$$

By definition, we deduce that f is Lipschitz, with Lipschitz constant 2.

(ii) By Picard's theorem (see Theorem 6 and its remark), there exists a global solution to the given system for all $t > 0$. In particular, we should not expect that the system exhibits finite-time blowup.

Example 12. Consider the system $\dot{x} = r - x - e^{-x}$ for $r \in \mathbb{R}$. Sketch all qualitatively different vector fields that occur as r is varied, determine the bifurcation point(s) (x^*, r) if it exists, identify the type of bifurcation, and sketch the bifurcation diagram of x^* against r.

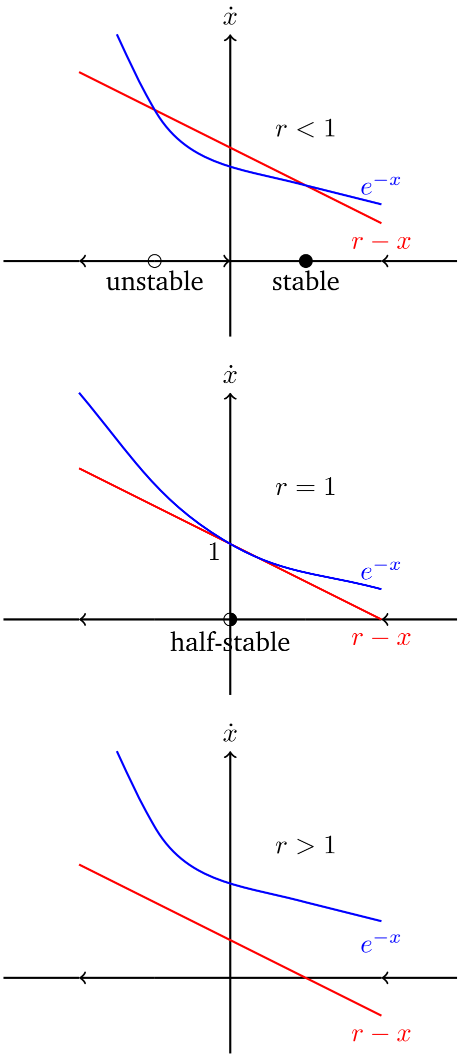
Suggested Solution:

Using the theorem of identification of possible bifurcation points, ie Theorem 9, we focus on points (x^*, r) that are solutions to $f = 0$ and $\frac{\partial}{\partial x} f = 0$. These correspond to

$$\begin{cases} r - x - e^{-x} & = 0, \\ -1 + e^{-x} & = 0. \end{cases}$$

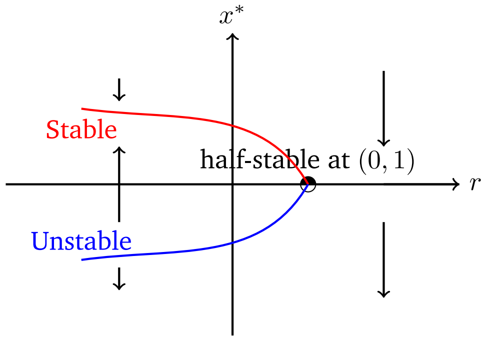
In this case, if one tries to solve the system, one should obtain the solution as $(x^*, r) = (0, 1)$. This prompts us to sketch phase portraits for $r < 1$, $r = 1$, and $r > 1$ to determine if it is an actual bifurcation point and the type of bifurcation.

However, note that it might be challenging to sketch $r - x - e^{-x}$ directly. Thus, one can turn to graphical methods for this. Instead, we consider sketching the graphs $r - x$ and e^{-x}. If the graph for $r - x$ is above e^{-x}, this implies that $\dot{x} = r - x - e^{-x} > 0$ and vice versa.
Hence, we have for $\dot{x} = r - x - e^{-x}$,



and thus $(x^*, r) = (0, 1)$ is indeed a bifurcation point and exhibits saddle-node bifurcation.

To sketch the bifurcation diagram, we plot a graph of x^* against r as a "collection of different phase portraits" as shown below:


x^*
Stable
half-stable at $(0, 1)$
r
Unstable

4 Discussion 4

Bifurcations II: Pitchfork Bifurcations and General Theory.

Consider a system

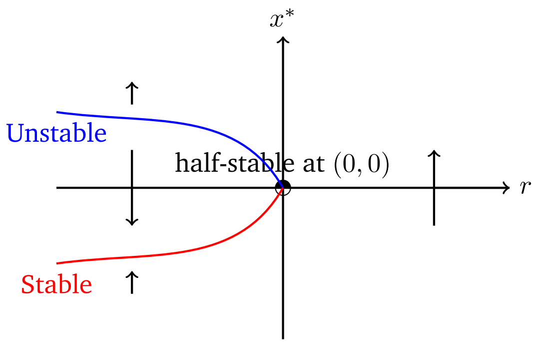
$$\dot{x} = f(x, r)$$

where r refers to a parameter of the system. Suppose that the system has a smooth curve of fixed points $(x, r) = (x^*(r), r)$ for r in some interval. If the stability of the fixed point $x^*(r)$ changes at some $r = r^*$, we say that the system undergoes **bifurcation** at $(x, r) = (x^*, r^*)$.

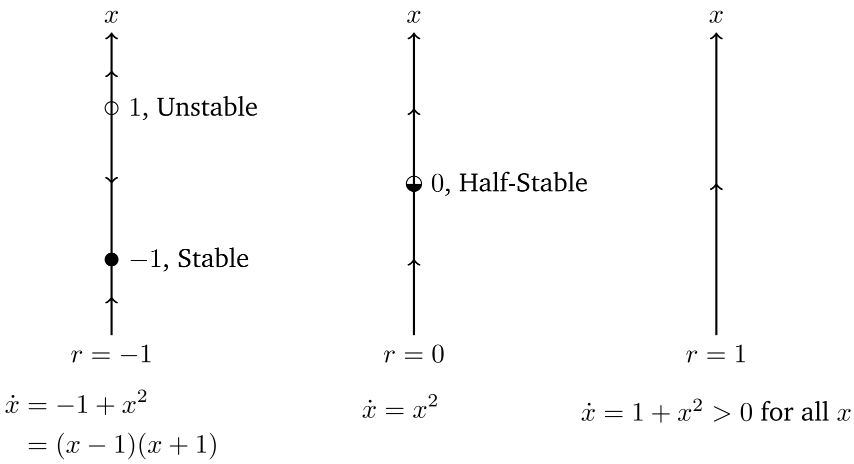
Roughly speaking, at some critical value of the parameter $r = r^*$, we see a change in the characteristics of the fixed point(s) denoted by x^*. These can be visualized on the $x^* - r$ plane, where r is the contentious parameter for bifurcation, in which for each value of r, we draw a phase portrait in x and stack them continuously while emphasizing the fixed points.

Combining with the two other types of bifurcations covered in the last discussion, here are three common types of bifurcations.

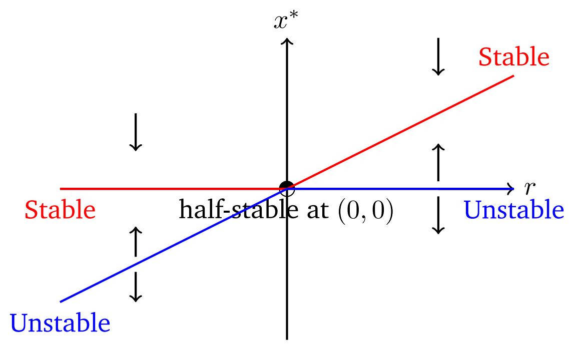
Saddle-Node: (Mutual destruction/creation of fixed points), for instance, $\dot{x} = r + x^2$.
Bifurcation point: $(x^*, r) = (0, 0)$.



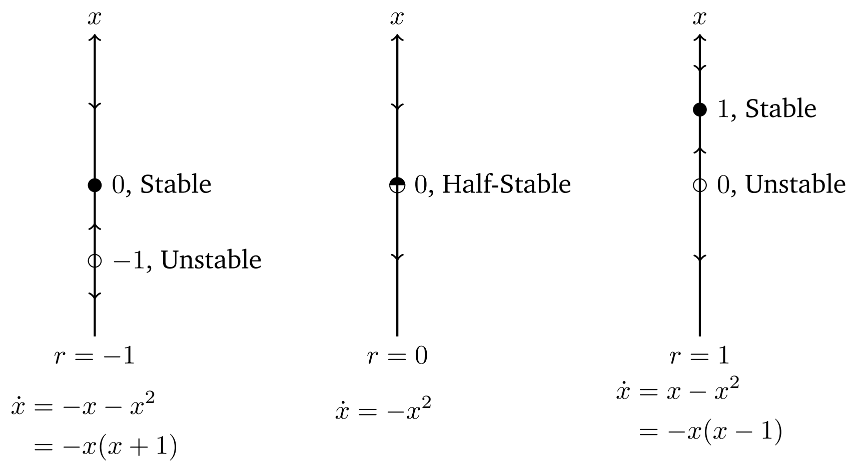
Indeed, to see that the above bifurcation diagram makes sense, we look at three key regimes ($r < 0$, $r = 0$, and $r > 0$) and draw the phase portraits vertically (so that we can observe how it gets stacked).



<u>Transcritical:</u> ("Transfer of critical points"), for instance, $\dot{x} = rx - x^2$.
Bifurcation point: $(x^*, r) = (0, 0)$.

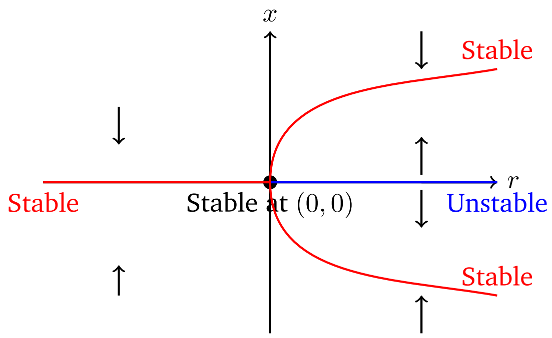


Indeed, to see that the above bifurcation diagram makes sense, we look at three key regimes ($r < 0$, $r = 0$, and $r > 0$) and draw the phase portraits vertically (so that we can observe how it gets stacked).

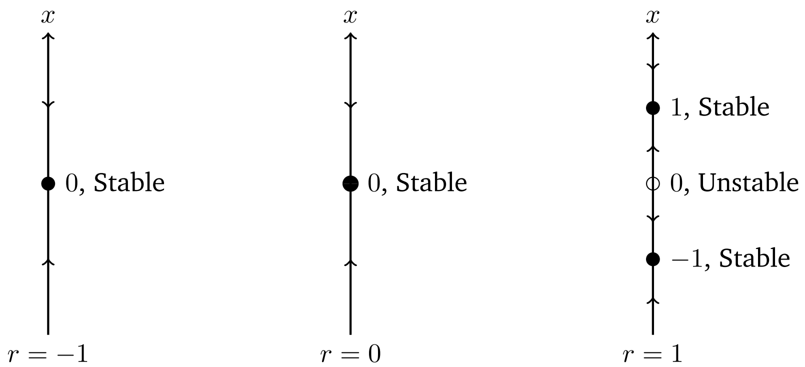


<u>Supercritical Pitchfork:</u>

Example: $\dot{x} = rx - x^3$.
Bifurcation point: $(x^*, r) = (0, 0)$.



Indeed, to see that the above bifurcation diagram makes sense, we look at three key regimes ($r < 0$, $r = 0$, and $r > 0$) and draw the phase portraits vertically (so that we can observe how it gets stacked).



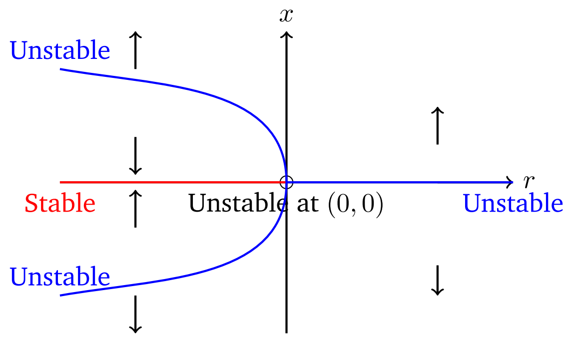
$\dot{x} = -x - x^3$
$= -x(1 + x^2)$
one root, $x = 0$

$\dot{x} = -x^3$

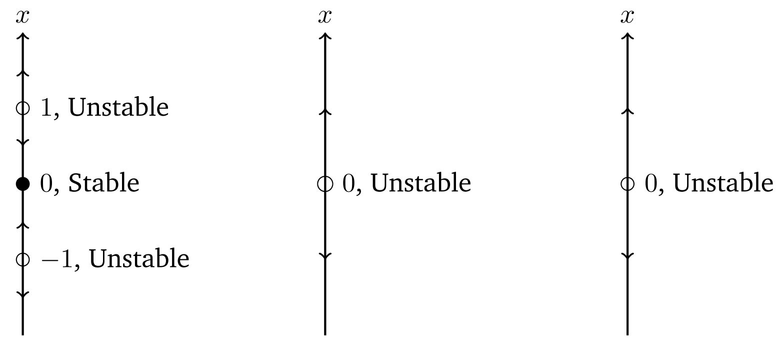
$\dot{x} = x - x^3$
$= -x(x-1)(x+1)$
three roots, $x = -1, 0, 1$

<u>Subcritical Pitchfork:</u>

Example: $\dot{x} = rx + x^3$.
Bifurcation point: $(x^*, r) = (0, 0)$.



Indeed, to see that the above bifurcation diagram makes sense, we look at three key regimes ($r < 0$, $r = 0$, and $r > 0$) and draw the phase portraits vertically (so that we can observe how it gets stacked).



$r = -1$

$$\dot{x} = -x + x^3 = x(x-1)(x+1)$$

three roots, $x = -1, 0, 1$

$r = 0$

$$\dot{x} = x^3$$

$r = 1$

$$\dot{x} = x + x^3 = x(1 + x^2) = x \times \text{“} > 0\text{”}$$

one root, $x = 0$

Remark: How I would remember these:

- Think of more stable branches as good, so super $\to$ “more stable”.
- Unstable outer fork - Subcritical.
  (Sub: More unstable branches before colliding - a suboptimal case.
  Alternatively, overall, we have more unstable branches.
  Indeed, we can check that there are 3 unstable branches and 1 stable branch in the subcritical pitchfork diagram above.)
- Stable outer fork - Supercritical.
  (Sup: More stable branches before colliding - a superoptimal case.
  Alternatively, overall, we have more stable branches. Indeed, we can check that there are 3 stable branches and 1 unstable branch in the supercritical pitchfork diagram above.)

The following theorem might help with the identification of bifurcation points (and serve as a "double-checking" mechanism for drawing bifurcation diagrams/used to flag potential bifurcation points.)

Theorem 13. (Identification of Possible Bifurcation Points.)
If the system

$$\dot{x} = f(x, r)$$

has a bifurcation point at $(x, r) = (x^*, r^*)$, then

$$f(x^*, r^*) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial x} f(x^*, r^*) = 0.$$

Intuitively, the first equation corresponds to solving the fixed point $f(x^*) = 0$. Furthermore, recall that $f'(x^*) < 0$ corresponds to a stable fixed point, while $f'(x^*) > 0$ corresponds to an unstable fixed point. Thus, any change in stability should potentially occur at this transition point where $f'(x^*) = 0$.[9]

Note that the converse might not necessarily hold. Thus, if we solve the above equations $f = 0$ and $\frac{\partial}{\partial x} f = 0$, we have contentious bifurcation points (x^*, r^*). To determine if these are actually bifurcation points, we will have to resort to graphing methods.

[9]Note that we wrote $f'(x^*)$ above since f is a function of one variable, x. If we introduce the bifurcation parameter r, we now have $\dot{x} = f(x, r)$, where f is a function of two variables. Here, "$f'(x^*)$" now refers to differentiating with respect to x while keeping r fixed, which is precise the definition of the partial derivative $\frac{\partial f}{\partial r}(x, r)$!

The following are some examples related to the above concepts on bifurcations, and some concepts related to those covered in Discussions 2 and 3 (which would serve as some sort of review problems for the midterm).

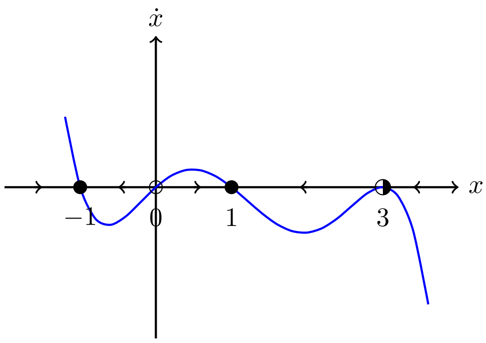
Example 14. Find a function $f(x)$ such that $\dot{x} = f(x)$ has **exactly** the following fixed points:

- A stable fixed point at -1,
- An unstable fixed point at 0,
- A stable fixed point at 1,
- A half-stable fixed point at 3,

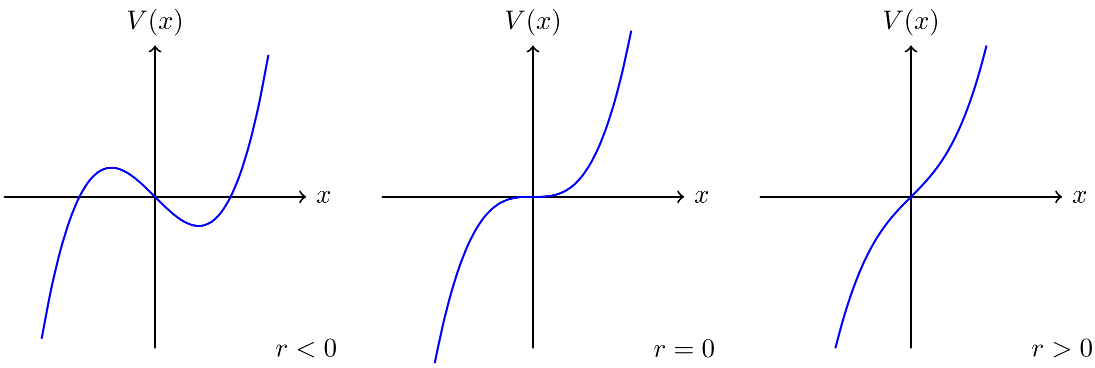
and draw the corresponding phase portrait.

Suggested Solution:

$$f(x) = -(x+1)x(x-1)(x-3)^2.$$



Example 15. A system $\dot{x} = f(x, r)$ has a bifurcation at $x = 0, r = 0$. A potential function for f has the following graphs:



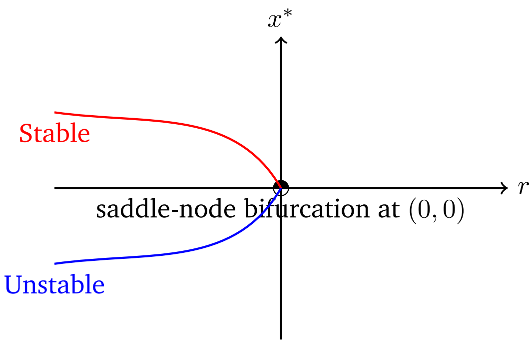
Sketch the bifurcation diagram and identify the type of bifurcation.

Suggested Solution:

By inspection, we have

- For $r < 0$, two fixed points, with the negative fixed point being unstable and the positive fixed point being stable.
- For $r = 0$, one fixed point, with the fixed point at zero being half-stable (stable from the right).
- For $r > 0$, no fixed points.

Consequently, we have a **saddle-node bifurcation** at $(x^*, r) = (0, 0)$. The bifurcation diagram is given by



Example 16. Consider the system

$$\begin{cases} \dot{x} = (x-1)^{\frac{1}{4}} \\ x(0) = 1. \end{cases}$$

(i) Find two different solutions to the system above.

(ii) Explain why your answer in (i) does not contradict Picard's theorem.

Suggested Solutions:

(i) One solution is given by $x_1(t) = 1$ for all time t. To find the second solution, we would have to solve the equation by separation of variables. This yields

$$\begin{aligned} \int \frac{\mathrm{d}x}{(x-1)^{\frac{1}{4}}} &= \int \mathrm{d}t \\ \frac{4}{3}(x-1)^{\frac{3}{4}} &= t + C. \end{aligned}$$

Using $x(0) = 1$, we have $C = 0$. Thus, we have for all $t \in \mathbb{R}$

$$x_2(t) = 1 + \left(\frac{3}{4}t\right)^{\frac{4}{3}}.$$

(ii) This does not contradicts Picard's theorem as $f(x) = (x-1)^{\frac{1}{4}}$ is not Lipschitz at $x = 1$, so solutions in the vicinity of $t = 0$ need not be unique.

(Alternatively, we know that a corollary of Picard's theorem is that if f is continuously differentiable, then we will have a unique local solution. We can instead easily check that $f(x) = (x-1)^{\frac{1}{4}}$ is not continuously differentiable at $x = 1$; in fact, it is not even differentiable!)

(Proof that it is indeed not Lipschitz at $x = 1$. Suppose for a contradiction that it is, then we must have $|f(x) - f(1)| \le L|x-1|$ for all $x \in \mathbb{R}$. Equivalently, we must have

$$\begin{aligned} |(x-1)^{\frac{1}{4}} - (1-1)^{\frac{1}{4}}| &\le L|x-1| \\ |x-1|^{\frac{1}{4}} &\le L|x-1| \\ \frac{1}{|x-1|^{\frac{3}{4}}} &\le L. \end{aligned}$$

Hence, if we pick x to be sufficiently close to 1 (i.e. $x \to 1$), we see that $|x-1| \to \infty$, contradicting that it must satisfy the above upper bound for all $x \in \mathbb{R}$.)

Example 17. Sketch a global bifurcation diagram for the equation

$$\dot{x} = (x^2 + r^2 - 1)(x^2 - 4).$$

Be sure to label all branches with their stability and classify all bifurcation points.

Suggested Solution: The following illustrates a series of steps that you should do if you encounter a problem that asks you to sketch a bifurcation diagram.

Step 1 - Identify all fixed points (as a function of r.)

Indeed, we have $f(x) = (x^2 + r^2 - 1)(x^2 - 4) = 0$ implies that $x^2 = 1 - r^2$ or $x^2 = 4$. The latter yields $x = \pm 2$ as the fixed points (regardless of the values of r). For the former, we observe the following phenomenon:

- $x^2 = 1 - r^2$ has no roots if $1 - r^2 < 0$, that is, $r^2 > 1$, or equivalently, $r > 1$ or $r < -1$.
- $x^2 = 1 - r^2$ has only one root if $1 - r^2 = 0$, that is, when $r = \pm 1$. The repeated root is $x = 0$.
- $x^2 = 1 - r^2$ has two roots if $1 - r^2 > 0$, that is, $r^2 < 1$, or equivalently, $-1 < r < 1$. We denote them by $x_- = -\sqrt{1 - r^2}$ and $x_+ = \sqrt{1 - r^2}$. The two roots are $x = \pm\sqrt{1 - r^2}$. In this case, the roots are always between -1 and 1, and will never overlap with the two other roots $x = \pm 2$ obtained from the other equation.

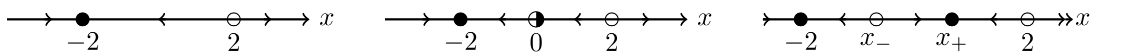
Step 2 - Flag all possible bifurcation points by analyzing the change in the number of fixed points.

From the above analysis, we see that $r = -1$ and $r = 1$ are possible bifurcation points (since that corresponds to the change in the number of roots to the equation $x^2 = 1 - r^2$, and hence possibly change in relevant stability).

Step 3 - For each possible bifurcation point, sketch phase portraits for parameters close to these points.

For $r = -1$, we see that

$(x^2 - 4)$:	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$+$
$(x^2 + r^2 - 1)$:	$+$	$-$	$+$	$+$	0	$+$	$+$	$+$	$-$	$+$	$+$
Overall Sign:	$+$	$-$	$+$	$+$	0	$+$	$+$	$-$	$+$	$-$	$+$

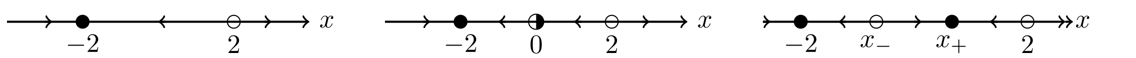


$r < -1$ $\qquad$ $r = -1$ $\qquad$ $1 > r > -1$

In the above, we are hinging on the fact that to draw a phase portrait, you really only need the sign of each of the terms contributing to the function $\dot{x} = f(x) = (x^2 - 4)(x^2 + r^2 - 1)$. From the above phase portraits, we can see that $x = 0, r = -1$ corresponds to a saddle-node bifurcation (ie two new critical points of opposite stability are created out of "nothing").

For $r = 1$ (read the diagram from right to left for increasing r), we see that

$(x^2 - 4)$:	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$+$
$(x^2 + r^2 - 1)$:	$+$	$-$	$+$	$+$	0	$+$	$+$	$+$	$-$	$+$	$+$
Overall Sign:	$+$	$-$	$+$	$+$	0	$+$	$+$	$-$	$+$	$-$	$+$



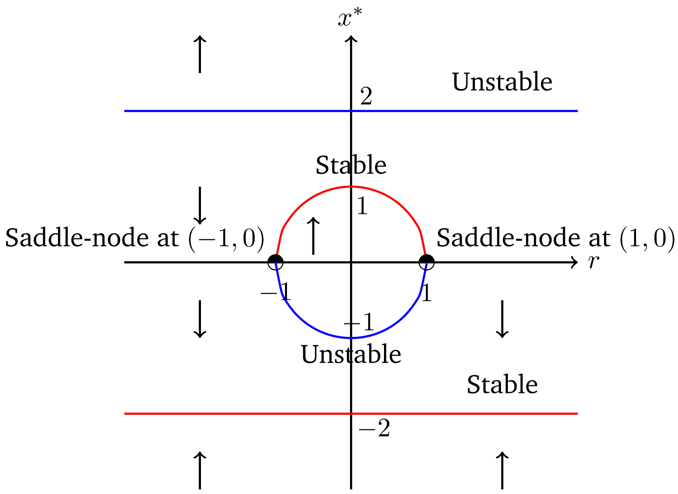
$r > 1$ $\qquad$ $r = 1$ $\qquad$ $1 > r > -1$

From the above phase portraits, we can see that $x = 0, r = 1$ corresponds to a saddle-node bifurcation (ie two critical points of opposite stability are annihilated).

Step 4: Identify curves of bifurcation.

From Step 1, we note that $x_* = \pm 2$ and $x_*^2 + r^2 - 1 = 0$. These are curves in which we will observe fixed points to fall onto. In fact, the latter equation can be written as $x_*^2 + r^2 = 1$, which corresponds to the equation of a circle centered at the origin with radius 1. The former corresponds to two lines $x_* = \pm 2$.

Step 5: Combined everything to draw the global bifurcation diagram.



Example 18. (Exercise 2.5.3 in Strogatz, Modified.)

(i) Consider the equation

$$\begin{cases} \dot{y} = y^3 \\ y(0) = y_0. \end{cases}$$

Show that the equation above exhibits finite time blow-up for any $y_0 > 0$.

(ii) Consider the equation

$$\begin{cases} \dot{x} = rx + x^3 \\ x(0) = x_0, \end{cases}$$

where $r > 0$ is fixed. Show that $|x(t)| \to \infty$ in finite time, starting from any initial condition $x_0 \neq 0$.

Suggested Solution:

(i) We do so by solving the equation by separation of variables. Indeed, this yields

$$-\frac{1}{y^2} = t + C.$$

Plugging $y(0) = 1$, we have $-1 = C$. This thus implies that

$$\frac{1}{y^2} = 1 - t.$$

Hence, we see that $y \to \infty$ if $\frac{1}{y^2} \to 0$, and this happens if $1 - t \to 0$, or in other words, $t \to 1$. Hence, we indeed observe that the above equation exhibits finite time blow-up.

(ii) Note that if $x_0 = x(0) > 0$, then $\dot{x}(0) = rx(0) + x(0)^3 > 0$ and by continuity of $rx + x^3$, x is increasing in an interval containing 0. We can repeat this argument to show that x is increasing for all $t \geq 0$ and thus $x(t) \geq x(0) > 0$. Thus, we can show that $rx + x^3 \geq x^3$. By (i) and comparison theorem (pick $y_0 = x_0$ so we start from the same initial data), this implies that $x(t) \geq y(t)$ for all t. In particular, at time T such that $y(t) \to \infty$ as $t \to T$, we then have $x(t) \to \infty$ as $t \to \infty$, and thus $x(t)$ blows up in finite time (ie $|x(t)| \to \infty$).

For $x_0 = x(0) < 0$, we note that $\dot{x}(0) = rx(0) + x(0)^3 < 0$ since $r > 0$. Define $z(t) = -x(t)$ (Intuition: blows up in the negative direction). Thus, we have

$$\begin{cases} \dot{z} = rz + z^3 \\ z(0) = -x_0, \end{cases}$$

since we have $\dot{z} = -\dot{x} = -rx - rx^3 = r(-x) + r(-x)^3 = rz + z^3$ with $z(0) = -x(0) = -x_0 > 0$. We repeat the exact same argument in the previous part to show that $z(t) \to \infty$ in finite time. Since $z(t) = -x(t)$, this implies that $x(t) \to -\infty$ in finite time. Nonetheless, we have $|x(t)| \to \infty$ and we are done.

5 Discussion 5

Non-dimensionalization.
In any physical (and/or social science) models, the corresponding (differential) equations tend to model certain (physical) phenomena. In the context of ODEs, a first order ODE has the following general form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y(x), x).$$

In the equation above, we notice that there are two types of variables, mainly

- Independent variable, x.
- Dependent variable (since it depends on x via the differential equation, the independent variable), given by y.

These physical variables tend to have their corresponding units. For instance, we can describe the velocity of an object with mass m falling through a viscous fluid with drag coefficient b from a point, with velocity $v(t)$ depending on time t via the following ODE:

$$mv'(t) = -mg + b(v(t))^2,$$

where g is the acceleration due to free-fall. It is important to identify which of the following are constants - in this case, m, b and g are constants (parameters). Here, we assume that these constants are positive to avoid technical difficulties. The dependent variable is given by v and the independent variable is given by time t. As both the dependent and the independent variables have physical units, this motivates us to consider the following non-dimensionalized units:

$$\tau = \frac{t}{T} \qquad \text{and} \qquad \nu(\tau) = \frac{v(\tau(t))}{V}.$$

Note that T and V are (possibly dimensional-ful) constants (such that τ and ν are dimensionless). Furthermore, note that **such a substitution is true for all T and V, and thus we have the freedom to choose T and V as we like.** Using the equations above, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}\tau} \qquad \text{and} \qquad \nu = \frac{v}{V}.$$

Plug them into our original equation, we have

$$\begin{aligned} m\frac{\mathrm{d}}{\mathrm{d}t}v(t) &= -mg + b(v(t))^2 \\ m\frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}\tau}\nu(\tau)V &= -mg + b(\nu(\tau)V)^2 \\ \frac{mV}{T}\nu'(\tau) &= -mg + bV^2\nu(\tau)^2 \end{aligned}$$

Note that we have two degrees of freedom (ie free to choose T and V). Furthermore, we can always divide the equation via a common constant to make one of the terms in the equation above dimensionless. This implies that we can have a total of 3 dimensionless terms, ie the equation above can be entirely dimensionless! Following such a logic, we first divide the entire equation by mg to obtain:

$$\frac{V}{gT}\nu'(\tau) = -1 + \frac{bV^2}{mg}\nu(\tau)^2.$$

Using my rights to choose T and V, I can pick them such that the coefficients in front of ν' and ν^2 are equals to 1. Equivalently, this implies that

$$\frac{bV^2}{mg} = 1 \to V = \sqrt{\frac{mg}{b}}$$

and

$$\frac{V}{gT} = 1 \to T = \frac{V}{g} = \sqrt{\frac{m}{bg}}.$$

The resultant equation is given by

$$\nu' = -1 + \nu^2,$$

with

$$\tau = \frac{t}{\sqrt{\frac{m}{bg}}} \text{ and } \nu = \frac{v}{\sqrt{\frac{mg}{b}}}.$$

As an added bonus, we will show that by doing whatever we have done, we get for free that these variables are dimensionless. For instance, from phyical considerations, we know that t has units [time], m has units [mass] and g has units [meter] $\times$ [second]$^{-2}$. For b, either we know the units or we can deduce the units from the equation. Since the original equation has $-mg$ added to bv^2, for two physical quantities to be added, they must be of the same units. This implies that units for $b = \frac{\text{units for m} \times \text{units for g}}{\text{units for v}^2} = \text{[mass][meter]}^{-1}$. Now, we can check that

$$\text{units for } \tau = \frac{\text{units for } t}{\sqrt{\frac{\text{units for } m}{\text{units for } b \times \text{units for } g}}} = \frac{\text{[second]}}{\frac{\text{[mass]}^{1/2}}{\text{[mass]}^{1/2}\text{[meter]}^{-1/2}\text{[meter]}^{1/2}\text{[second]}^{-1}}} = \text{no units.}$$

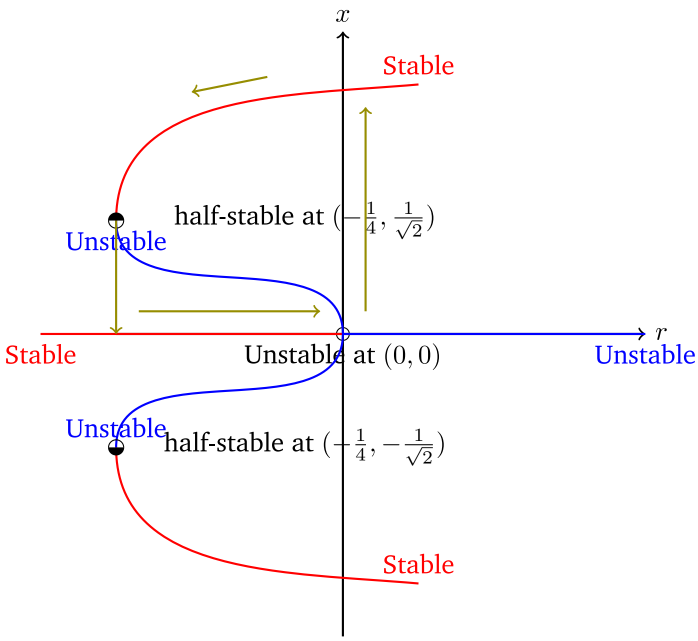
One can similarly check this for the other constant V.

In summary, we can always:

- Divide an equation through to make one of the constants dimensionless.
- Free to pick any scaling parameters that we use in our substitution to match what we need (or to render them dimensionless)!

We shall look at an example of this in a bit.

Hysteresis:
(Increasing the value of the parameter, followed by decreasing it back to its original value, leads to different trajectories.)
Example: $\dot{x} = rx + x^3 - x^5$. The hysteresis is shown in the olive trajectory. As we cross to the origin from the left (ie $r < 0$, close to the origin and then increase the parameter by a little), the trajectory jumps up to the stable branch. Hence, we would have to decrease the parameter by a significant factor (ie till we have $r < -1/4$), and then increase the parameter back up to where we start from to return back to the original value, instead of just decreasing the parameter by the amount that we have increased by.



Imperfect Bifurcation.

We consider another general form of an autonomous first-order system for $x(t)$ with two independent parameters r, h as given below:

$$\dot{x} = f(x, r, h).$$

In previous examples, we have always labeled r as a parameter. Sometimes, one can distinguish h here by giving h a name, namely the **imperfection parameter**.

If we are given a fixed r or h, we can treat the other parameter as the bifurcation parameter, and sketch a bifurcation diagram as per usual. However, if we attempt to vary both parameters together, it is more informative to sketch the **stability diagram**, a diagram for h against r (or the other way round). In this diagram, we should observe that

- To sketch such a diagram, we first compute the equation of the fixed points as a function of both r and h. Then, we observe the "curves" of r and h such that we observe a change in the number of fixed points. (This is similar to what we would usually do for a bifurcation diagram, instead for two variables!)
- Crossing the curves usually corresponds to changing the number of fixed points corresponding to the type of bifurcation (ie transcritical does not change the number of fixed points, except at the critical value in which both fixed points combine into a half-stable point, etc).
- Such a diagram should be labeled with the number of fixed points in each of the regions separated by the curves.

We shall look at an example below.

Example 19. (Strogatz Exercise 3.5.7, Modified). Consider the logistic equation modelling the population $N(t)$ as follows:

$$\begin{cases} \dot{N}(t) = rN\left(1 - \frac{N}{K}\right), \\ N(0) = N_0. \end{cases}$$

Show that the system can be re-written in the dimensionless form:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}\tau} = x(1-x), \\ x(0) = x_0. \end{cases}$$

for appropriate choices of the dimensionless variable x, x_0 and τ.

Suggested Solutions:

Set $x = \frac{N}{M}$, and $\tau = \frac{t}{T}$ for some M and T to be determined and that each of the x and τ are dimensionless. Recall that

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}\tau}.$$

Substitute these into the equation, we have:

$$\begin{cases} \frac{M}{T}\frac{\mathrm{d}x}{\mathrm{d}\tau} = rMx(1 - \frac{M}{K}x) \\ x(0) = \frac{N_0}{M}. \end{cases}$$

which simplifies to:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}\tau} = rTx(1 - \frac{M}{K}x) \\ x(0) = \frac{N_0}{M}. \end{cases} .$$

To match the required equations, we set $rT = 1$ and $\frac{M}{K} = 1$, ie

$$T = \frac{1}{r} \qquad \text{and} \qquad K = M$$

and thus

$$\tau = rt \qquad \text{and} \qquad x = \frac{N}{K}.$$

The system thus reduces to

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}\tau} = x(1-x) \\ x(0) = x_0, \end{cases}$$

with

$$x_0 = \frac{N_0}{M} = \frac{N_0}{K}.$$

Example 20. (Strogatz 3.6.3 Modified, A perturbation to the supercritical pitchfork.)
Consider the system

$$\dot{x} = rx + ax^2 - x^3$$

where $a \in \mathbb{R}$. When $a = 0$, we have the normal form for the supercritical pitchfork. The goal of this exercise is to study the effects of the new parameter a.

(i) For each a, there is a bifurcation diagram of x^* against r. As a varies, these bifurcation diagrams can undergo qualitative changes. Sketch all the qualitatively different bifurcation diagrams that can be obtained by varying a.

(ii) Sketch the corresponding stability diagram in the (r, a) plane.

Suggested Solutions:

Step 1: Identify all fixed points (as a function of r and a). We start off by solving $f = 0$. This implies that

$$rx + ax^2 - x^3 = x(r + ax - x^2) = 0.$$

This yields

$$x = 0 \quad \text{or} \quad x = \frac{1}{2}\left(a \pm \sqrt{a^2 + 4r}\right).$$

The roots obtained from the second solution can be analyzed as follows:

- No roots if $r < -\frac{a^2}{4}$,
- Repeated roots if $r = -\frac{a^2}{4}$, and
- Two roots if $r > -\frac{a^2}{4}$.

(This should remind you of a saddle-node bifurcation, but now this is along the curve $r = -\frac{a^2}{4}$.)

Step 2: Flag possible bifurcation curves by analyzing the change in the number of fixed points.
We have already pointed out that $r = -\frac{a^2}{4}$ is one such curve. Next, we observe if the roots could ever overlap. This involves solving

$$0 = \frac{1}{2}\left(a \pm \sqrt{a^2 + 4r}\right).$$

Moving a to the other side and squaring, we see that

$$a^2 = a^2 + 4r$$

or just

$$r = 0.$$

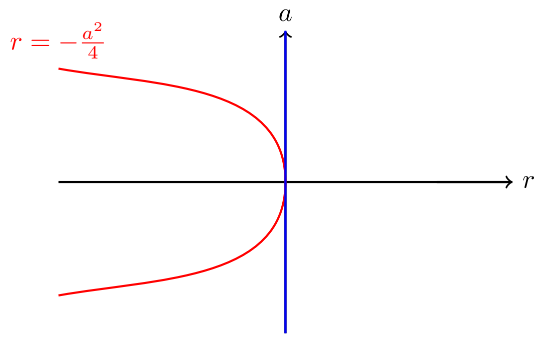
This implies that along $r = 0$ (for any a, recall that these are "curves", not points), we might have roots overlapping, which might correspond to a potential bifurcation curve.

Step 3: Sketch the bifurcation curves on a stability diagram.

Remark: This step comes **before** sketching any phase portraits for the following reasons:

- It is sometimes not clear if we should be allowing r to vary first, and sketch bifurcation diagrams for $x^* - a$, or to allow a to vary first and sketch bifurcation diagrams for $x^* - r$.
- It is easier to fill in different regions on the stability diagram on the go as required than to treat r and a as two arbitrary parameters, in which the algebra will get challenging. In fact, once you have determined the relevant region, we can substitute actual numbers to simplify our computations.
- In exams, the final product is usually somewhat of the form of obtaining the stability diagram (though possibly guided). Hence, sketching the curve directly shows that you have made progress on the problem (even though you have not sketched any phase portraits!)

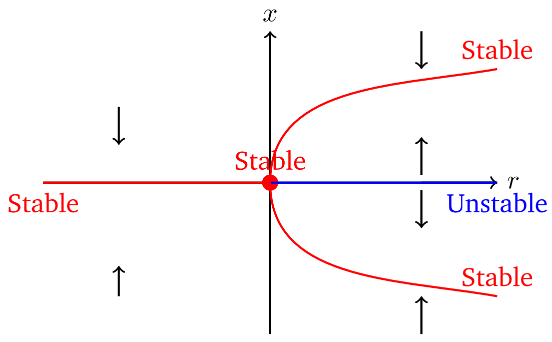
Hence, we have



where the colored curves correspond to potential bifurcation curves as determined in the second step.

Step 4: For each fixed a or r, sketch a bifurcation diagram.
As observed from the diagram above and by the prompts given in the problem, we should be sketching bifurcation diagrams for $x^* - r$ for a given a (that could vary). Furthermore, since $a = 0$ corresponds to the normal form of a supercritical pitchfork, it makes sense to look at regions for which $a < 0, a = 0$, and $a > 0$. In fact, *a priori* from the bifurcation curves above, we expect similar behaviors for $a < 0$ and $a > 0$ (since each horizontal line always cuts the red curve and the blue line, except at $a = 0$).

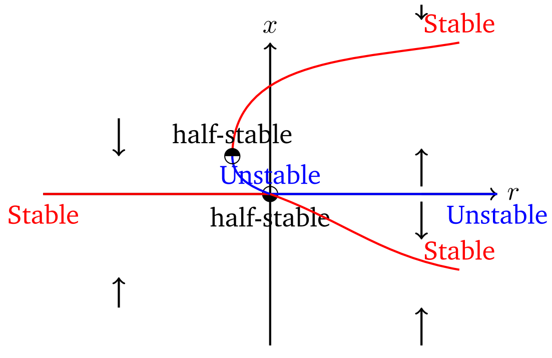
For $a = 0$, we have



This implies

# Fixed Points	$r < 0$	$r = 0$	$r > 0$
	1	1	3

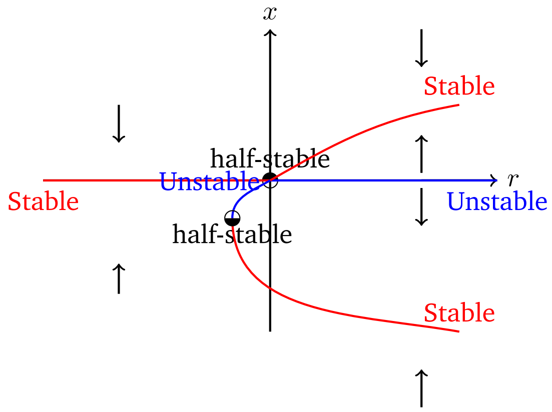
For $a > 0$, we have



This implies

# Fixed Points	$r < -a^2/4$	$r = -a^2/4$	$-a^2/4 < r < 0$	$r = 0$	$r > 0$
	1	2	3	2	3

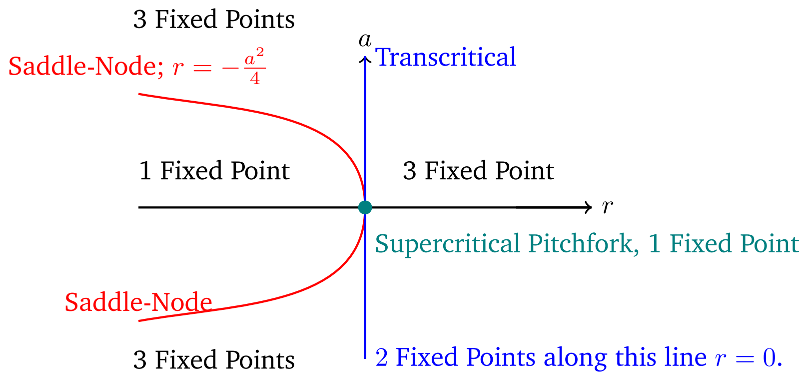
For $a < 0$, we get



This implies

# Fixed Points	$r < -a^2/4$	$r = -a^2/4$	$-a^2/4 < r < 0$	$r = 0$	$r > 0$
	1	2	3	2	3

<u>Step 5: Combine everything to draw the global stability diagram.</u>



6 Discussion 6

Flows on a Circle.

For flows on a circle, we are generally looking at an autonomous first-order system in the form of

$$\dot{\theta} = f(\theta)$$

where $\theta \in (-\pi, \pi]$.[10] In general, we demand that f must be 2π-periodic (i.e $f(\theta) = f(\theta + 2\pi)$ for any θ) for this to make sense. As this is an autonomous first-order system, we can import various definitions on flows on a line to this case here.
For flows in a circle, some of the common terminologies include:

- If $\dot{\theta} = \omega$, we say that ω is the (angular) **frequency** of the system.
- In general, we define the **period**, T, to be the time taken to complete one loop. Mathematically, the "formula" is given by:
$$T = \int_0^{2\pi} \frac{\mathrm{d}t}{\mathrm{d}\theta} \mathrm{d}\theta = \int_0^{2\pi} \frac{1}{\dot{\theta}(\theta)} \mathrm{d}\theta.$$

Similar to the one-dimensional flows on a line, we can have bifurcation occurring in our system. A bifurcation analysis for flows on a circle is analogous to that for flows on a line (with the extra precaution that we are only interested in $\theta \in (-\pi, \pi]$).

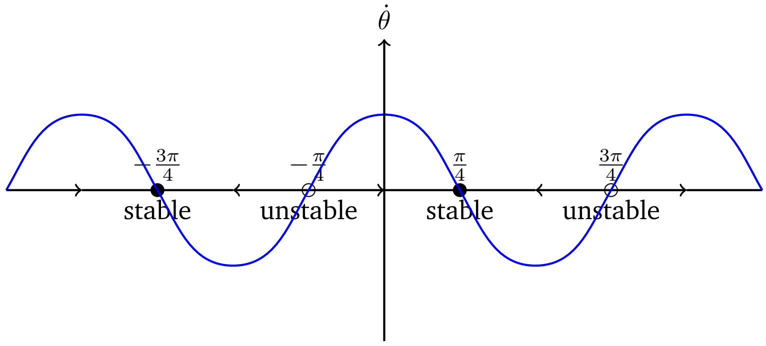
We shall look at more examples this week, including examples of flows on a circle, bifurcation, and evaluating relevant periods.

[10] It does not matter if you include $-\pi$ or not, as the behavior at $-\pi$ must be identical to that for π since f is 2π-periodic.

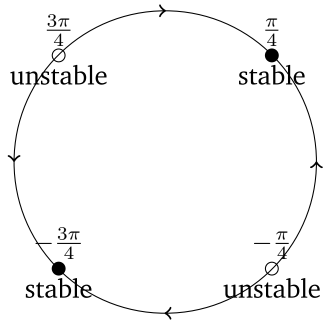
Example 21. Draw a phase portrait for

$$\dot{\theta} = \cos(2\theta).$$

Suggested Solutions:
For this system, we can solve for the fixed points to be $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ by solving $\dot{\theta} = 0$. We then sketch the graph of $\dot{\theta}$ against θ on a line.



Equivalently, in a circle, we have



Example 22. (Strogatz 4.5.3 Modified) Consider the following system:

$$\dot{\theta} = \mu + \sin(\theta).$$

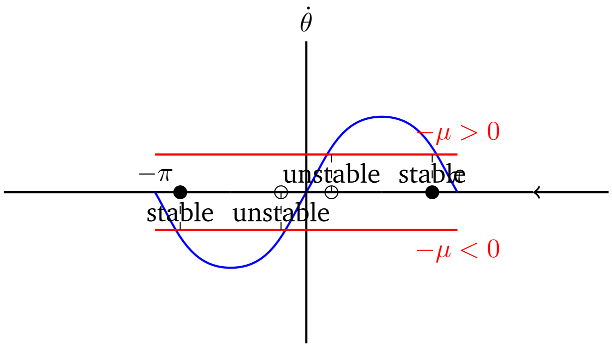
(i) Draw all qualitatively different phase portraits for different values of the control parameter μ. Classify the bifurcations that occur as μ varies, and find all the bifurcation values of μ.

(ii) For $\mu > 1$, write down the expression representing the period of oscillation for the associated flow.

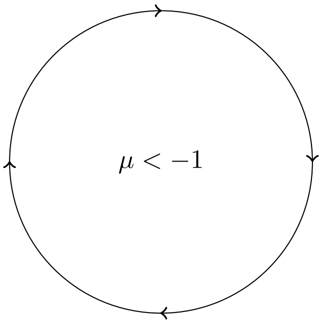
Suggested Solutions:
(i) Let $f(\theta, \mu) = \sin(\theta) - (-\mu)$. Graphically, if we view $f(\theta, \mu) = \sin(\theta) - (-\mu)$, we see that as $\mu \in (-1, 1)$, we obtain two roots to the equation $f = 0$. Indeed, from the graph below, we see that

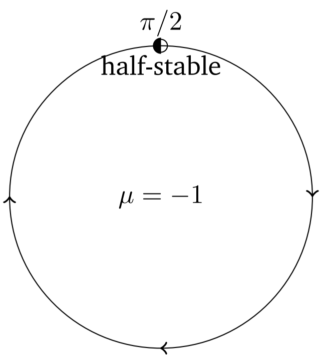
- For $1 > -\mu > 0$ (or $\mu \in (-1, 0)$), we have the roots to be given by $\arcsin|\mu|$ and $\pi - \arcsin(|\mu|)$.
- For $-1 < -\mu < 0$ (or $\mu \in (0, 1)$), we have the roots to be given by $-\arcsin(|\mu|)$ and $-\pi + \arcsin(|\mu|)$.
- At $\mu = 1$ (or $-\mu = -1$), we see that the two roots combine to give a single root at $\theta = \frac{\pi}{2}$.
- Similarly, at $\mu = -1$ (or $-\mu = 1$), we see that the two roots combine to give a single root at $\theta = -\frac{\pi}{2}$.

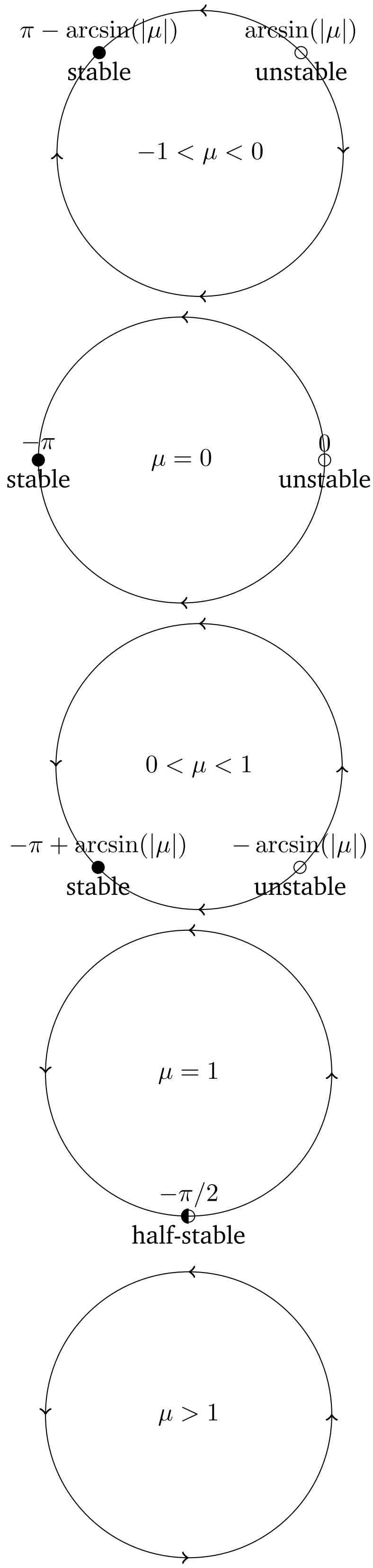
Hence, we have that the saddle-node bifurcation occurs at $(\mu, \theta) = (\pm 1, \mp\frac{\pi}{2})$.



The phase portraits are thus given as follows:







From the phase portraits, we can see that both bifurcation points correspond to the saddle-node bifurcation.

(ii) The period of the oscillation is given by

$$T = \int_0^{2\pi} \frac{1}{\dot{\theta}} \mathrm{d}\theta = \int_0^{2\pi} \frac{1}{\mu + \sin\theta} \mathrm{d}\theta.$$

Example 23. In the lectures, we have explored the idea of looking at a square root scaling law for the time taken for a trajectory to pass through a bottleneck. In general, this was obtained for a saddle-node bifurcation by computing

$$T_{\text{bottleneck}} \approx \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{\dot{x}(x)}$$

where $\dot{x} = r + x^2$ for $r > 0$. This formula is valid as for r is small, most of the time is spent near the bottleneck as $|\dot{x}|$ is close to 0. Similarly, let us consider

$$\dot{x} = \cosh(x) - 1 + r.$$

One can see that for $r > 0$, the graph of $\dot{x}$ against x has a minimum value at $x = 0, \dot{x} = r$.

(i) Following a similar argument, write down the integral representing the time spent $T(r)$ by a trajectory in a bottleneck for small values of r.

(ii) Compute the integral numerically for $r = 0.001$ and $r = 0.0001$ using your favourite numerical solver (i.e Wolfram etc), and thus compute the ratio

$$\frac{T(0.0001)}{T(0.001)}.$$

(iii) How does your solution in (b) to verify that the square root scaling law works in this case?

(iv) Why should you expect the square root scaling law to work in this case?

Suggested Solutions:
(i)

$$T(r) \approx \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{\cosh(x) - 1 + r}.$$

(ii) We have

$$\frac{T(0.0001)}{T(0.001)} \approx 3.19279.$$

(iii) Since

$$\left(\frac{T(0.0001)}{T(0.001)}\right)^2 \approx 10.1939 \approx 10,$$

we can see that scaling the factor a down by a factor of 10 causes the time taken to scale up by a factor of $\approx \sqrt{10}$.
(iv) Observe that for small x, we have $\cosh(x) \approx 1 + \frac{x^2}{2}$, and thus we have

$$\dot{x} \approx x^2 + r,$$

corresponding to the normal form of a saddle-node bifurcation.

7 Discussion 7

Recap for Required Linear Algebra
Properties of 2×2 matrices.
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary matrix and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be the identity matrix.

- **Trace** of the matrix A is defined and denoted by

$$\mathrm{tr}(A) = a + d.$$

- **Determinant** of the matrix A is defined and denoted by

$$\det(A) = ad - bc.$$

- The **eigenvalue(s)** of the matrix A, denoted by λ, is/are obtained from solving the **characteristic polynomial** in λ given by

$$\det(A - \lambda I) = 0.$$

- We say that a vector $\mathbf{v} \neq 0$ is an **eigenvector** of A with an associated **eigenvalue** λ if

$$A\mathbf{v} = \lambda \mathbf{v}.$$

- We say that two vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ are **linearly independent** over $\mathbb{C}$ (or $\mathbb{R}$ if the eigenvalues are both real) if the following condition holds: If we have

$$C_1\mathbf{v_1} + C_2\mathbf{v_2} = 0$$

  for some undetermined $C_1, C_2 \in \mathbb{C}$, then we must have $C_1 = 0$ and $C_2 = 0$.

- **Distinct eigenvalues correspond to distinct eigenvectors.** If A has two distinct eigenvalues $\lambda_1 \neq \lambda_2$, then the corresponding eigenvectors are linearly independent.

We also have the following important theorem that classifies the **canonical form** of 2×2 matrices as follows.

Theorem 24. Let A be a 2×2 matrix. Then we are in one of the three situations:

- A has linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with corresponding eigenvalues λ_1 and λ_2. Then, taking $P = [\mathbf{v}_1, \mathbf{v}_2]$, we have

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

- A has complex eigenvectors $\mathbf{v} \pm \mathrm{i}\mathbf{w}$ with corresponding complex eigenvalues $\gamma \pm \mathrm{i}\omega$. Then, taking $P = [\mathbf{v}, \mathbf{w}]$, we have

$$P^{-1}AP = \begin{bmatrix} \gamma & \omega \\ -\omega & \gamma \end{bmatrix}.$$

  Remark: This is of a slightly different form than in the homework (with $-\omega$ in the first row, and ω in the second row); I've chosen to preserve this form as this follows from the right choice of basis (ie P).

- A has one real eigenvector $\mathbf{v}$ with repeated eigenvalue σ. We can write down the generalized eigenvector $\mathbf{w}$ as one that solves $A\mathbf{w} = \sigma\mathbf{w} + \mathbf{v}$. Then, taking $P = [\mathbf{v}, \mathbf{w}]$, we have

$$P^{-1}AP = \begin{bmatrix} \sigma & 1 \\ 0 & \sigma \end{bmatrix}.$$

2×2 Linear Systems

Consider the 2D system of the form

$$\dot{\mathbf{x}} = A\mathbf{x}$$

for 2×2 matrices. Note that

$$\mathbf{x} = \mathbf{0}$$

is a fixed point since $\dot{\mathbf{x}} = \mathbf{0}$ if $\mathbf{x} = \mathbf{0}$.
To analyze such systems, note that we can transform the system such that the coefficient matrix is in one of the three canonical forms as described above. Mathematically, this can be done as follows. Let P be the corresponding invertible matrix P from the theorem above such that we have one of the three canonical forms.

Consider the substitution

$$\mathbf{y} = P^{-1}\mathbf{x} \iff \mathbf{x} = P\mathbf{y}.$$

This implies that

$$\begin{aligned}\dot{\mathbf{y}} &= P^{-1}\dot{\mathbf{x}}\\ &= P^{-1}A\mathbf{x}\\ &= P^{-1}AP\mathbf{y}.\end{aligned}$$

Since our choice of P is such that $P^{-1}AP$ is one of the three canonical forms, then we are looking at

$$\dot{\mathbf{y}} = (\text{Canonical Form Matrix})\,\mathbf{y}.$$

With that, it is sufficient to analyze the system for which the coefficient matrix is in its canonical form. Thus, without loss of generality, we assume that A is in its canonical form.

Classification of Fixed Points for 2×2 Linear Systems.

Distinct Eigenvalues - 2 Eigenvectors

We consider a system $\dot{\mathbf{x}} = A\mathbf{x}$, with A having eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are distinct in the sense that one is not a scalar multiple of the other.

Solution:

$$\boxed{\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2}.$$

Example: Take $\mathbf{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$.

Case 1: $\lambda_1 < \lambda_2 < 0$. For example, take $\lambda_1 = -2$ and $\lambda_2 = -1$. The solution is thus given by

$$\mathbf{x}(t) = C_1 e^{-2t}\mathbf{v}_1 + C_2 e^{-t}\mathbf{v}_2.$$

Observations:

- In both directions, e^{-t} and e^{-2t} goes to 0 as t goes to infinity.
- For $t \to +\infty$, we have $e^{-2t} \ll e^{-t}$. This implies that the dominant trajectory is along $\mathbf{v}_2$ (e^{-t}) for large t (ie close to $\mathbf{0}$ since the trajectories go to $\mathbf{0}$).
- For $t \to -\infty$, we have $e^{-2t} \gg e^{-t}$. Thus, the trajectories start off being parallel to $\mathbf{v}_1$.
- The fixed point $(0, 0)$ is also known as a *stable* node.

Case 2: $\lambda_1 > \lambda_2 > 0$. For example, take $\lambda_1 = 2$ and $\lambda_2 = 1$. The solution is thus given by

$$\mathbf{x}(t) = C_1 e^{2t}\mathbf{v}_1 + C_2 e^{t}\mathbf{v}_2.$$

Observations:

- In both directions, e^{t} and e^{2t} goes to ∞ as t goes to infinity.
- For $t \to \infty$, we have $e^{2t} \gg e^{t}$. This implies that the dominant trajectory is along $\mathbf{v}_1$ (e^{2t}) for large t (ie the trajectories will turn to be almost parallel to $\mathbf{v}_1$).
- For $t \to -\infty$, we have $e^{2t} \ll e^{t}$. The trajectories thus starts off being parallel to $\mathbf{v}_2$ (e^{t}).
- The fixed point $(0, 0)$ is also known as an *unstable* node.

Case 3: $\lambda_1 < 0 < \lambda_2$. For example, take $\lambda_1 = -1$ and $\lambda_2 = 1$. The solution is thus given by

$$\mathbf{x}(t) = C_1 e^{-t}\mathbf{v}_1 + C_2 e^{t}\mathbf{v}_2.$$

Observations:

- Along the $\mathbf{v}_1$ direction, e^{-t} goes to 0 as $t \to \infty$. We label $\text{span}\{\mathbf{v}_1\}$ as the *stable* manifold.
- Along the $\mathbf{v}_2$ direction, e^{t} goes to $+\infty$ as $t \to \infty$. We label $\text{span}\{\mathbf{v}_2\}$ as the *unstable* manifold.
- For $t \to \infty$, the dominant trajectory is along $\mathbf{v_2}$. This implies that trajectories turn to be parallel $\mathbf{v}_2$. This trajectory also approaches $\mathbf{v}_2$ since the coefficient in front of $\mathbf{v}_1$ goes to 0 (that is, e^{-t} goes to 0).
- For $t \to -\infty$, the dominant trajectory is $\mathbf{v}_1$, and thus starts being parallel to $\mathbf{v}_1$.
- The fixed point $(0, 0)$ is also known as an *saddle* node.

Case 4: $\lambda_1 = 0, \lambda_2 \neq 0$ For example, take $\lambda_1 = 0$ and $\lambda_2 = -1$. The solution is thus given by

$$\mathbf{x}(t) = C_1\mathbf{v}_1 + C_2 e^{-t}\mathbf{v}_2.$$

Observations:

- Note that in the $\mathbf{v}_2$ direction, the coefficient e^{-t} goes to 0. Thus, all trajectory will lose their component in $\mathbf{v}_2$, and ultimately lie along $\text{span}\{\mathbf{v}_1\}$ as $t \to \infty$.
- Such a case is also known as a *line of fixed points*.

Repeated Eigenvalues

Case 5: A is diagonalizable. In other words, we can set $\sigma = \lambda_1 = \lambda_2$ in the case of distinct eigenvalues. The solution is thus given by

$$\mathbf{x}(t) = C_1 e^{\sigma t}\mathbf{v}_1 + C_2 e^{\sigma t}\mathbf{v}_2 = e^{\sigma t}\left(C_1\mathbf{v}_1 + C_2\mathbf{v}_2\right).$$

Observations:

- Without loss of generality, we assume that $\sigma \neq 0$ (else it reduces to the zero matrix).
- From the factorization above, starting from any point $\mathbf{x}$ (in which C_1 and C_2 are determined), the entire vector does not change in direction but instead, scales with $e^{\sigma t}$.
- Thus, if $\sigma > 0$, as $t \to \infty$, the trajectories thus goes to ∞ in $\mathbb{R}^2$. For $t \to -\infty$, this implies that $\mathbf{x} \to \mathbf{0}$.
- For $\sigma < 0$, we just have to reverse the direction of the trajectories.
- We say that $(0, 0)$ is an *unstable star* node for $\sigma > 0$ and a *stable star* node for $\sigma < 0$.

Case 6: A is in its Jordan Block. Suppose we have the Jordan block for $A = \begin{bmatrix} \sigma & 1 \\ 0 & \sigma \end{bmatrix}$, and let σ be the eigenvalue, $\mathbf{v}$ be the eigenvector, and $\mathbf{w}$ be the generalized eigenvector that solves $(A - \sigma I)\mathbf{w} = \mathbf{v}$.
Solution:

$$\boxed{\mathbf{x}(t) = (C_1 + C_2 t)e^{\sigma t}\mathbf{v} + C_2 e^{\sigma t}\mathbf{w}}$$

Example: Take $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
Observations:

- Let $C_2 > 0, \sigma > 0$. For $t \to \infty$, we have $te^{\sigma t}$ being the most dominant term. This implies that the trajectory will turn and be parallel to $\mathbf{v}$.
- For $t \to -\infty$, the dominant terms remains to be $te^{\sigma t}$ (since $|te^{\sigma t}| \gg |e^{\sigma t}|$ for $t \to -\infty$). With the additional negative sign from t, this implies that the trajectory starts off in the $-\mathbf{v}$ direction from the origin.
- Let $C_2 < 0, \sigma > 0$. Using a similar argument, we see that the trajectory instead starts off in the $+\mathbf{v}$ direction and ends off to be parallel to the $-\mathbf{v}$ direction as it goes to infinity.

- For $t \to -\infty$, the dominant terms remains to be $te^{\sigma t}$ (since $|te^{\sigma t}| \gg |e^{\sigma t}|$ for $t \to -\infty$). With the additional negative sign from t, this implies that the trajectory starts off in the $-\mathbf{v}$ direction if $C_2 > 0$.
- Note that since A is given, we can compute $\dot{\mathbf{x}} = A\mathbf{x}$ at a given point. Combining with the fact that trajectories go from $-\mathbf{v}$ to $\mathbf{v}$, we can pick the right direction (ie if it is clockwise or counterclockwise) in which the spiral should go.
- For $\sigma < 0$, we just have to reverse the direction of the trajectories.
- We say that $(0, 0)$ is an *unstable degenerate* node for $\sigma > 0$ and a *stable degenerate* node for $\sigma < 0$.

Case 7: A is in its Jordan Block and $\sigma = 0$.
Using the general solution and taking $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we have

$$\boxed{\mathbf{x}(t) = (C_1 + C_2 t)\mathbf{v} + C_2\mathbf{w}}.$$

One key piece of detail is that both C_2 appear in the coefficient of $\mathbf{v}$ and that of $\mathbf{w}$.
It suffices to consider the following three cases:

- Suppose $\mathbf{x}(0) = \mathbf{v} + \mathbf{w}$. Then, we have from the general solution that this is given by $C_1\mathbf{v} + C_2\mathbf{w}$, which implies that $C_1 = C_2 = 1$ necessarily. The general solution is given by $\mathbf{x}(t) = (1 + t)\mathbf{v} + \mathbf{w}$. As $t \to \infty$, we have $\mathbf{x} \to \infty\mathbf{v} + \mathbf{w}$, ie with one component of $\mathbf{w}$ and infinitely many components of $\mathbf{v}$ (going to infinity). Similarly, as $t \to -\infty$, we have $\mathbf{x}(0) = -\infty\mathbf{v} + \mathbf{w}$. From $t \to -\infty$, to 0 to ∞, we have

$$-\infty\mathbf{v} - \mathbf{w} \to -\mathbf{w} \to \infty\mathbf{v} - \mathbf{w}.$$

- Suppose $\mathbf{x}(0) = \mathbf{v}$. Then, we have from the general solution that this is given by $C_1\mathbf{v} + C_2\mathbf{w}$, which implies that $C_1 = 1, C_2 = 0$ necessarily. The general solution is given by $\mathbf{x}(t) = \mathbf{v}$. (Note that C_2 is coupled in both components of $\mathbf{v}$ and $\mathbf{w}$). Hence, the solution stays where it starts from for all time. This would be equivalent to the line of fixed points as in Case 4.
- Suppose $\mathbf{x}(0) = \mathbf{v} - \mathbf{w}$. Then, we have from the general solution that this is given by $C_1\mathbf{v} + C_2\mathbf{w}$, which implies that $C_1 = 1, C_2 = -1$ necessarily. The general solution is given by $\mathbf{x}(t) = (1 - t)\mathbf{v} - \mathbf{w}$. From $t \to -\infty$, to 0 to ∞, we have

$$\infty\mathbf{v} - \mathbf{w} \to -\mathbf{w} \to -\infty\mathbf{v} - \mathbf{w}.$$

Complex Eigenvalues.

Suppose we have the real canonical form of $A = \begin{bmatrix} \gamma & \omega \\ -\omega & \gamma \end{bmatrix}$, for some $\gamma, \omega \in \mathbb{R}$. Let the (real) eigenvectors be given by $\mathbf{v} \pm i\mathbf{w}$, with eigenvalues $\gamma \pm i\omega$.
Solution:

$$\boxed{\mathbf{x}(t) = Ae^{\gamma t}\sin(\omega t + B)\mathbf{v} + Ae^{\gamma t}\cos(\omega t + B)\mathbf{w}}.$$

Example: Take $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
Observations:

- Here, $e^{\gamma t}$ determines if the trajectories tend to $\mathbf{0}$ or go to ∞.
- Case 8: $\gamma \neq 0$. (Implicitly, we are also assuming that $\omega \neq 0$. This is because if that is not the case, this reduces to Case 5.) If $\gamma > 0$, the former happens, while if $\gamma < 0$, the latter happens (with trajectories reversed).
We say that $\mathbf{0}$ is an *unstable spiral* if $\alpha > 0$ and a *stable spiral* if $\alpha < 0$.
- Case 9: $\gamma = 0$. (Implicitly, we are assuming $\omega \neq 0$.) If $\gamma = 0$, we have that the exponential scaling factor $e^{\gamma t}$ disappears. Thus, trajectories go in circles with a constant radius from the fixed point. In this case, we call $\mathbf{0}$ a (linear) *center*.
- Without loss of generality, we set $B = 0$. Then, from the general solution

$$\mathbf{x}(t) = A\sin(\omega t)\mathbf{v} + B\cos(\omega t)\mathbf{w},$$

we see that the trajectory goes

$$\mathbf{w} \to \mathbf{v} \to -\mathbf{w} \to -\mathbf{v} \to \mathbf{w}$$

in one period. In other words, we can view the vector subtended by $\mathbf{v}$ and $\mathbf{w}$ as the semi-major/minor axes of an ellipse.

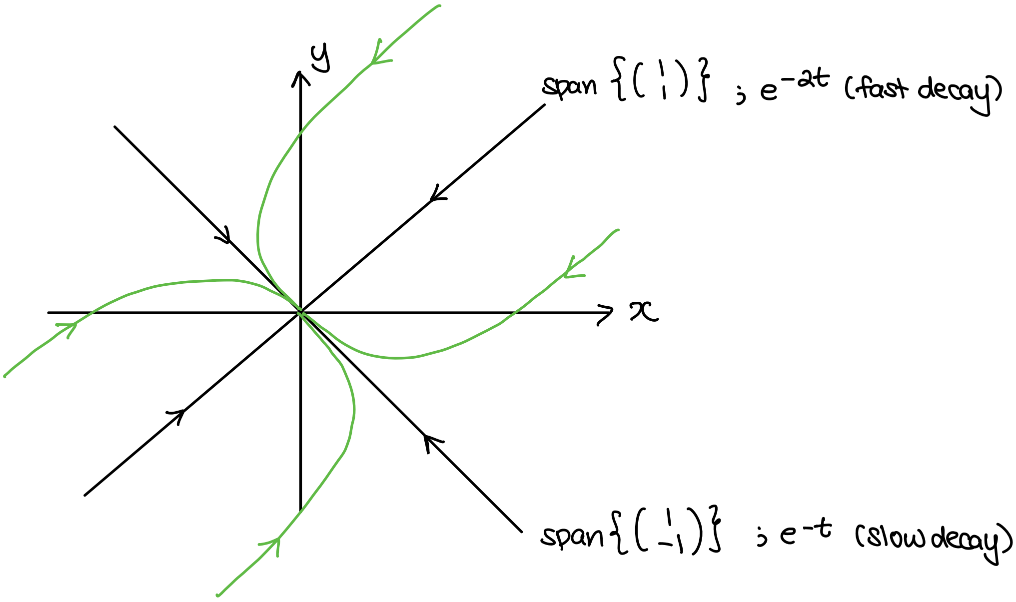
- To determine the actual direction of rotation, use the fact that $\dot{\mathbf{x}} = A\mathbf{x}$ to compute the direction at one point to determine the direction of "rotation" along the ellipse.

In summary, the "phase portrait strategies" are paired with each phase portrait for each case below.

Case 1. $(\lambda_1 < \lambda_2 < 0)$

$\lambda_1 = -2, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



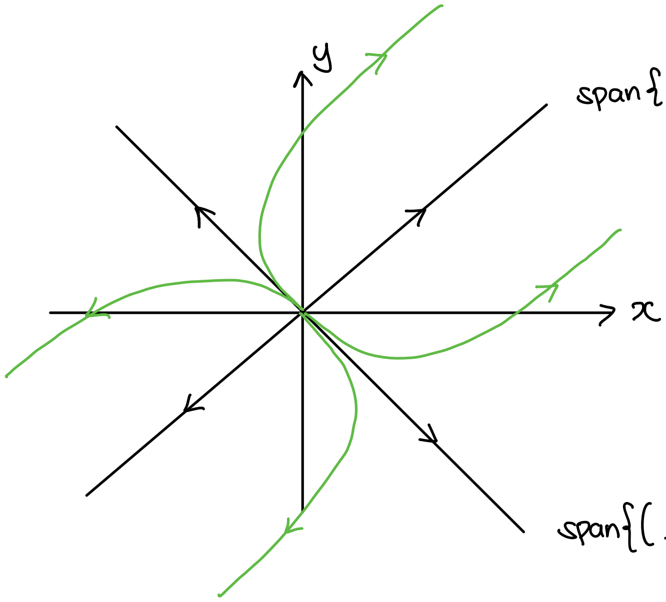
// to fast $\longrightarrow$ // to slow

(0,0) is a stable node.

Case 2. $(\lambda_1 > \lambda_2 > 0)$

$\lambda_1 = 2, \quad \vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 1, \quad \vec{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



$\text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$; e^{2t} (fast growth)

$\text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$; e^{t} (slow growth)

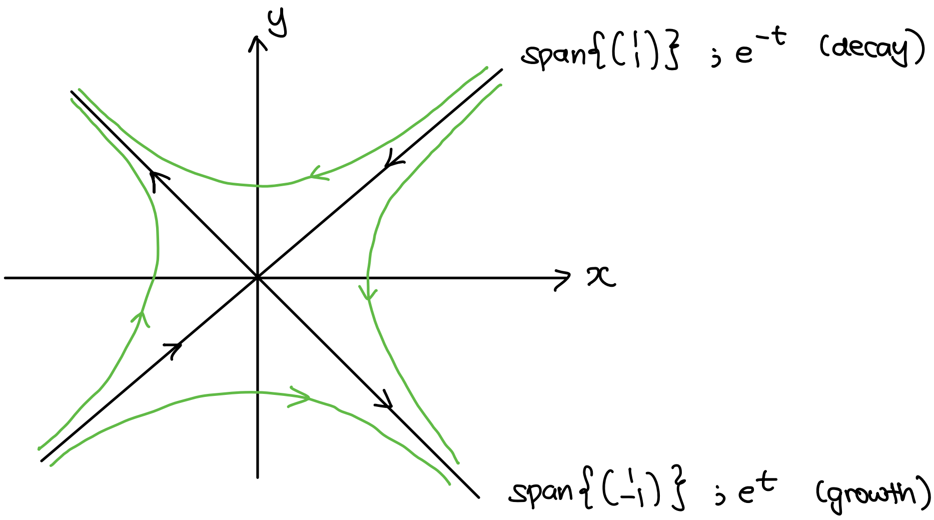
// to slow $\longrightarrow$ // to fast

(0,0) is an unstable node.

Case 3 $(\lambda_1 < 0 < \lambda_2)$

$\lambda_1 = -1, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 1, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



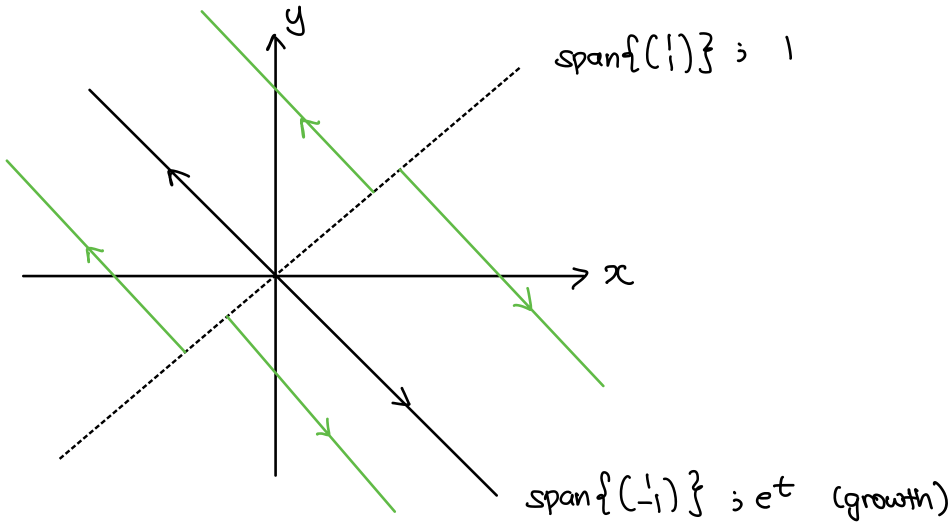
Follow parallel to eigendirections.

(0,0) is a saddle node.

Case 4 $(\lambda_1 = 0, \lambda_2 \neq 0)$

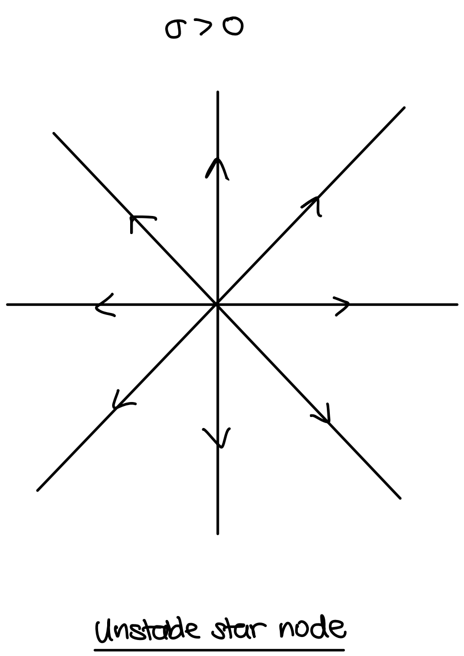
$\lambda_1 = 0, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

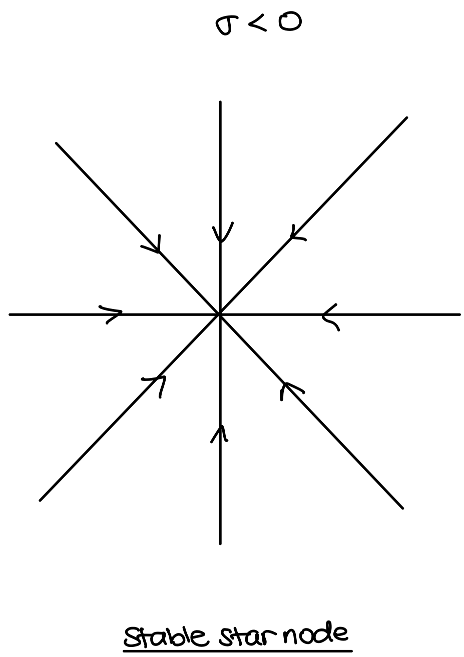
$\lambda_2 = -1, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



<u>Line of fixed points</u>

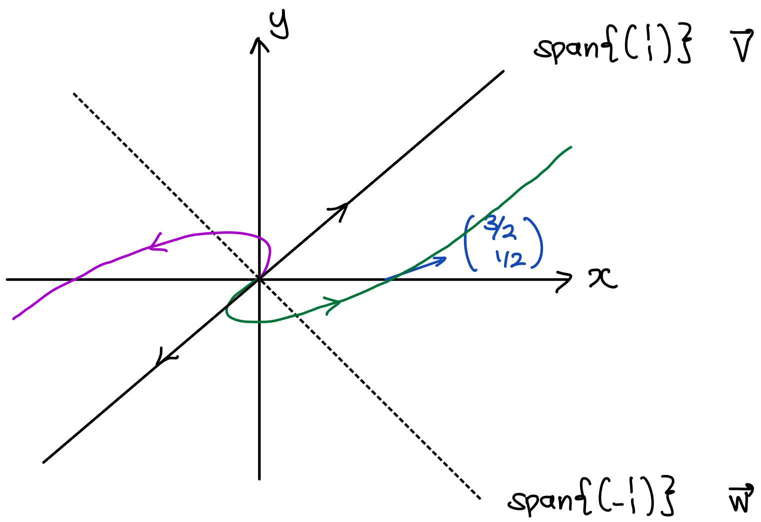
Case 5 $(\sigma \neq 0)$





Case 6

$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and $\sigma = 1$ (>0, wlog)



Unstable degenerate node

Step 1: Use $\dot{x} = Ax$ to guide you at a point

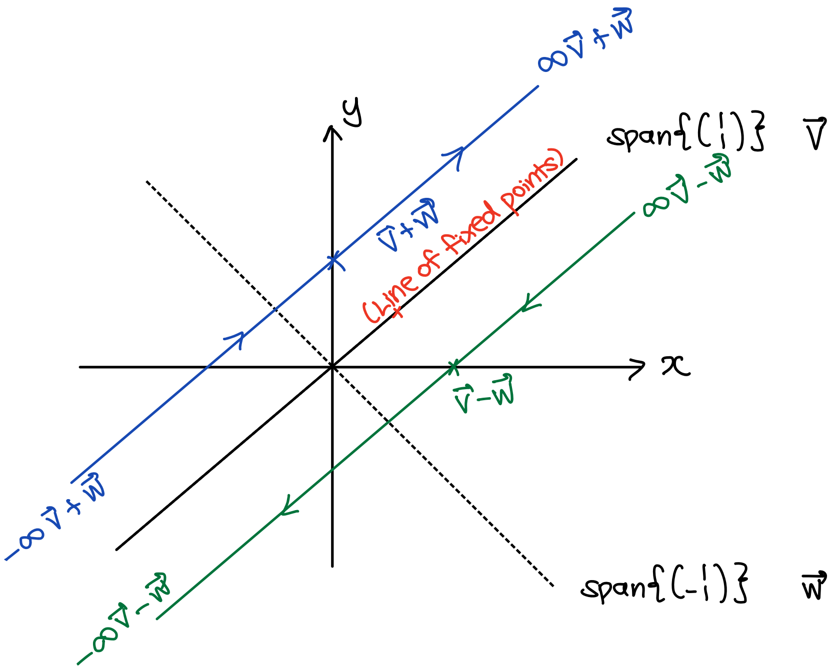
$A = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $\dot{x}$ at $(1,0)$ $= \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$

Step 2: $-\vec{v}$ to $\vec{v}$ (ie start along one direction, and be antiparallel

Step 3: Complete the diagram on the other side.

Case 7

$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and $\sigma = 0$



Line of fixed points.

$x(0) = \vec{v} + \vec{w}$ case

$x(0) = \vec{v}$ case

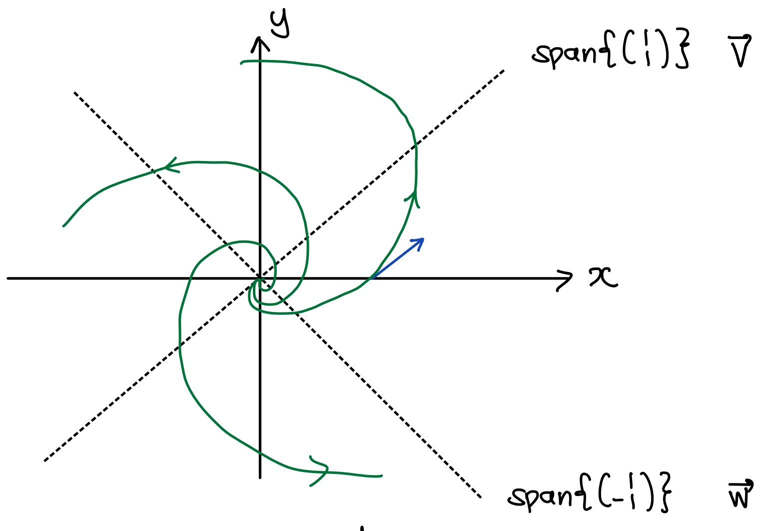
$x(0) = \vec{v} - \vec{w}$ case

Case 8

$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\lambda = 1 \pm i$

$\left(\begin{matrix} \gamma = 1, \ \omega = 1 \\ \gamma > 0 \ \text{wlog} \end{matrix} \right)$

Corresponding $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$



Unstable spiral

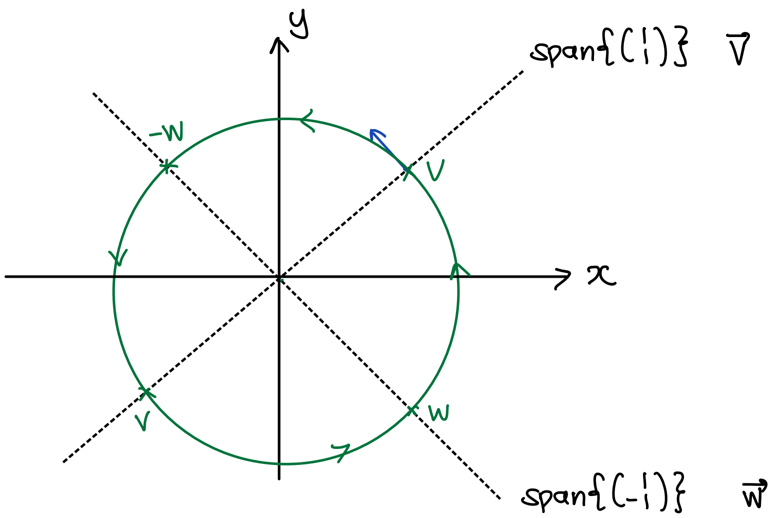
Since we know that it spirals outwards, it remains to determine vector field at any arbitrary point.

$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\dot{x}$ at $(1,0)$ $= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Case 9 $(\gamma = 0)$

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda = \pm i$$

Corresponding $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



Linear center

Step 1: Determine semi-major/minor axes.

Include $\vec{v}, \vec{w}, -\vec{v}, -\vec{w}$.

Complete the ellipse.

Step 2: Use $\dot{x} = Ax$ to guide you at a point

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \underset{\text{at } (1,1)}{\dot{x}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Example 25.

(i) Given the following matrix
$$A = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix},$$
find an invertible matrix P such that
$$P^{-1}AP$$
is in one of the three canonical forms of 2×2 matrices and write down the corresponding canonical form.

(ii) Hence or otherwise, plot the phase portrait of the following system. You should also classify the stability of the fixed point $(0,0)$ and indicate the eigenvectors in your sketch if the eigenvectors are real.
$$\begin{cases} \dot{x} = 7x + y, \\ \dot{y} = -4x + 3y. \end{cases}$$

Suggested Solution:
(i) Eigenvalues are given by solving $\det(A - \lambda I) = 0$. This yields the characteristic polynomial $(\lambda - 5)^2 = 0$. This implies that we only have $\lambda = 5$ as the only eigenvalue. Looking at the matrix for $A - 5I = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$. This implies that there is only one eigenvector given by
$$\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$
Next, we would like to be searching for the generalized eigenvector, that is, $\mathbf{w}$ that lies in $\ker((A-5I)^2)$ but not $\ker(A - 5I)$. This is done by solving[11]
$$(A - 5I)\mathbf{w} = \mathbf{v},$$
or equivalently,
$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \mathbf{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$
This can be done via simple row-reduction techniques, reducing the augmented matrix to its row-reduced form:
$$\begin{bmatrix} 2 & 1 & 1 \\ -4 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$
This implies that if $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, then we have
$$w_1 + \frac{1}{2}w_2 = \frac{1}{2}.$$
As long as the above equation is satisfied, we are done. One can observe that $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ does the job. This implies that we can pick
$$P = \begin{bmatrix} \mathbf{v} | \mathbf{w} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$
and obtain[12]
$$P^{-1}AP = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}.$$

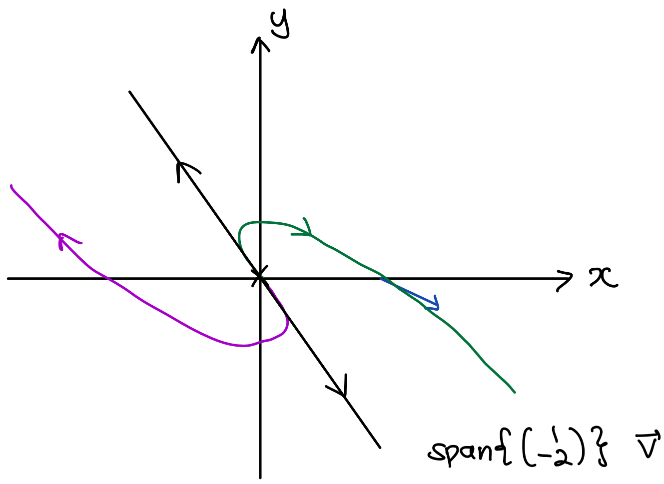
[11] Since this implies $(A - 5I)^2\mathbf{w} = (A - 5I)\mathbf{v} = \mathbf{0}$, where the second equality comes from the fact that we have solved for $\mathbf{v}$ from the equation $(A - 5I)\mathbf{v} = \mathbf{0}$.

[12] This is because now, we have $(A - 5\lambda I)\mathbf{w} = \mathbf{v}$, in which $A\mathbf{w} = \mathbf{v} + 5\mathbf{w}$, giving the column $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

(ii)

$\vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (only the eigenvector mattered)

with $\lambda = 5$.



Step 1: Use $\dot{x} = Ax$ to guide you at a point

$$A = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}, \quad \dot{x}_{\text{at } (1,0)} = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$$

Step 2: $-\vec{v}$ to $\vec{v}$ (ie start along one direction, and be antiparallel

Step 3: Complete the diagram on the other side.

(0,0) is an unstable degenerate node.

Example 26. In this example, you will attempt to derive a general form of some arbitrary system $\dot{\mathbf{x}} = A\mathbf{x}$, where A is not yet in its canonical form.

(i) Consider the system $\dot{\mathbf{y}} = B\mathbf{y}$, where $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ for $\lambda \neq \mu \neq 0$. Derive the general solution for $\mathbf{y}(t)$.

(ii) Suppose that A has two distinct non-zero eigenvalues given by λ and μ, with the associated eigenvectors $\mathbf{v}$ and $\mathbf{w}$. Derive the general solution to $\dot{\mathbf{x}} = A\mathbf{x}$ for $\mathbf{x}(t)$.

Suggested Solution:

(i) Let $\mathbf{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$. By comparing the first and second components, we have

$$\dot{y_1}(t) = \lambda y_1(t), \quad \dot{y_2}(t) = \mu y_2(t).$$

Solving each case by separation of variables, we have

$$y_1(t) = y_1(0)e^{\lambda t}, \quad y_2(t) = y_2(0)e^{\mu t}.$$

By treating $y_1(0)$ and $y_2(0)$ as arbitrary constants (depending on the initial data), we can write the above solution as

$$\mathbf{y}(t) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

(ii) If A is not in its canonical form, let $P = [\mathbf{v}\ \mathbf{w}]$ (which is invertible by Theorem 24). Perform the substitution

$$\mathbf{x} = P\mathbf{y} \iff \mathbf{y} = P^{-1}\mathbf{x}.$$

This implies that

$$P\dot{\mathbf{y}} = AP\mathbf{y}$$

and hence

$$\dot{\mathbf{y}} = P^{-1}AP\mathbf{y} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \mathbf{y}.$$

Using our answer in (i), we have

$$\mathbf{y}(t) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Plugging our substitution back in, we have

$$P^{-1}\mathbf{x}(t) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

and hence

$$\mathbf{x}(t) = P \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\mathbf{v}\ \mathbf{w}] \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

for some arbitrary constants c_1 and c_2.

8 Discussion 8

2×2 nonlinear systems.

Consider the 2D system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

with $\mathbf{x} \in \mathbb{R}^2$. Writing them out as vectors, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}.$$

We say that $\mathbf{x}^* \in \mathbb{R}^2$ is a **fixed point** of the system if $\mathbf{f}(\mathbf{x}^*) = 0$.

We can also obtain a corresponding linearized system about the fixed point by considering the multivariate Taylor's theorem as follows. Set $\mathbf{x}(t) = \mathbf{x}^* + \boldsymbol{\eta}(t)$. This implies that $\dot{\mathbf{x}}(t) = \dot{\boldsymbol{\eta}}(t)$. Furthermore, we can expand $\mathbf{f}(\mathbf{x})$ about $\mathbf{x}^*$ to obtain:

$$\dot{\boldsymbol{\eta}}(t) = \dot{\boldsymbol{x}}(t) = \mathbf{f}(\mathbf{x}(t)) = \mathbf{f}(\mathbf{x}^*) + (\nabla\mathbf{f})(\mathbf{x}^*)(\boldsymbol{\eta}(t)) + O(|\boldsymbol{\eta}|^2).$$

Since $\mathbf{f}(\mathbf{x}^*) = 0$ by definition of a fixed point, we have

$$\dot{\boldsymbol{\eta}}(t) \approx \nabla\mathbf{f}(\mathbf{x}^*) \cdot \boldsymbol{\eta}(t)$$

where $\nabla\mathbf{f}(\mathbf{x}^*)$ represents the Jacobian of $\mathbf{f}$ evaluated at the point $\mathbf{x}^*$ with the formula given by

$$\nabla\mathbf{f}(\mathbf{x}^*) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} (\mathbf{x}^*).$$

Note that $\nabla\mathbf{f}(\mathbf{x}^*)$ is a matrix with constant entries. Thus, it makes sense for us to view systems in the form of

$$\dot{\boldsymbol{\eta}} = \nabla\mathbf{f}(\mathbf{x}^*)\boldsymbol{\eta}$$

with the relevant linearized dynamical system centered at the fixed point $\mathbf{x} = \mathbf{x}^*$. In fact, the following theorem allows us to "transfer" the behavior of most behavior of linearized systems to the nonlinear system.

Theorem 27. (Hartman-Grobman Theorem) Let $\mathbf{f}$ be smooth and $\mathbf{x}^*$ be a hyperbolic fixed point of

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

Then, there is a continuously differentiable change of variable from a neighborhood of the fixed point $\mathbf{x}^*$ to a neighborhood of the fixed point $\boldsymbol{\eta}^* = 0$ with the linearized flow:

$$\dot{\boldsymbol{\eta}} = \nabla\mathbf{f}(\mathbf{x}^*)\boldsymbol{\eta}.$$

In summary, the above theorem says that

- If the real part of **all** the eigenvalues of $\nabla\mathbf{f}(\mathbf{x}^*)$ are non-zero, then the phase portrait for such a fixed point (called a **hyperbolic fixed point**) can be used to sketch the phase portrait for the original nonlinear system close to the fixed point $\mathbf{x} = \mathbf{x}^*$.
- **Excluded cases:**
  - Linearized Center (ie eigenvalues are $0 \pm i\omega$).
  - Distinct eigenvalues with at least one of them attaining zero. (ie eigenvalue $\lambda > \mu$, and one of them is zero).
  - Repeated eigenvalues with the zero eigenvalues repeated. ($\sigma = 0$ or $\lambda = \mu = 0$).

To analyze the dynamics for these excluded cases, we have to resort to alternative techniques, ie analyzing the system in polar coordination, finding invariant, using nullclines, etc.

Nullclines and Phase Portraits for 2×2 nonlinear systems

Consider the system

$$\begin{cases} \dot{x} = f_1(x, y), \\ \dot{y} = f_2(x, y). \end{cases}$$

- If we set $\dot{x} = f_1(x, y) = 0$, we (might) obtain some function g such that $y = g(x)$.
- Along such a curve, we know that $\dot{x} = 0$.
- Thus, all vector fields will only point in the vertical ($y-$) direction.
- We call the function $y = g(x)$ the $x-$*nullcline* (as the function does not (null) "climb" in the x direction).
- Similarly, if we set $\dot{y} = f_2(x, y) = 0$, along the curve in which $f_2(x, y) = 0$, the vector field will only point in the horizontal ($x-$) direction.
- The corresponding direction of the vector field along the nullclines can be determined by looking at the corresponding equation. For instance, for $x-$nullclines, since $\dot{x} = 0$, the vector fields point in the y direction. This direction can thus be determined from the sign of $\dot{y}$ from its corresponding equation.
- The curve given by $f_2(x, y) = 0$ is known as the $y-$*nullcline*.

The above concept becomes clearer if we look at a specific example in a bit.

Combining all of the above, given a nonlinear system

$$\begin{cases} \dot{x} = f_1(x, y), \\ \dot{y} = f_2(x, y), \end{cases}$$

here are the following steps that one should do if asked to sketch the phase portrait of it.

1. Find the fixed points of the system; ie solve $f_1(x^*, y^*) = 0$ and $f_2(x^*, y^*) = 0$.
2. Find the linearized system at each fixed point by computing the Jacobian $\nabla \mathbf{f}$ at each of the fixed points (x^*, y^*). Recall that
$$\nabla \mathbf{f}(x^*, y^*) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} (x^*, y^*).$$
3. Find the corresponding eigenvalues and eigenvectors of the Jacobian $\nabla \mathbf{f}(x^*, y^*)$.

   Now, we can start sketching the phase portrait as follows:
4. Start by plotting each of the fixed points on the xy plane. For each of the **hyperbolic** fixed points, sketch the phase portrait around the fixed points as if it was the linear system $\dot{\boldsymbol{\eta}} = \nabla \mathbf{f}(x^*, y^*)\boldsymbol{\eta}$ with the matrix $A = \nabla \mathbf{f}(x^*, y^*)$ (and using the eigenvalues and eigenvectors etc).
5. For each non-hyperbolic fixed point, this should be guided from parts of the question. Such analysis includes looking in polar coordinates, looking for invariants, etc.
6. Last but not least, sketch the $x-$ and $y-$ nullclines and indicate the direction of vector fields along each of these nullclines. If the nullclines have a complicated functional form, it might be fine to not include it in the final sketch of the phase portrait. (Grading for this portion is highly dependent on the grader and the explicit requirement(s) of a given question.)
7. All in all, try to complete the phase portrait by joining up whatever curves/vector fields that make sense.

We shall look at a full example.

Example 28. Consider the following system of ODEs:

$$\begin{cases} \dot{x} = y - 2, \\ \dot{y} = 1 - x. \end{cases}$$

(i) Find all $x-$ and $y-$ nullcline(s) to the above system.

(ii) Using only your answer in (i), deduce that $(1, 2)$ is the only fixed point to the above system.

(iii) By only using your answer in (i) (ie without finding eigenvectors and eigenvalues of the corresponding matrix A), sketch a phase portrait for the above system as if $(1, 2)$ is a linear center.

Suggested Solutions:

(i) The x-nullclines are given by $\dot{x} = 0$. In other words, we have

$$y - 2 = 0$$

and hence

$$y = 2.$$

Similarly, the y-nullclines are given by $\dot{y} = 0$. In other words, we have

$$1 - x = 0$$

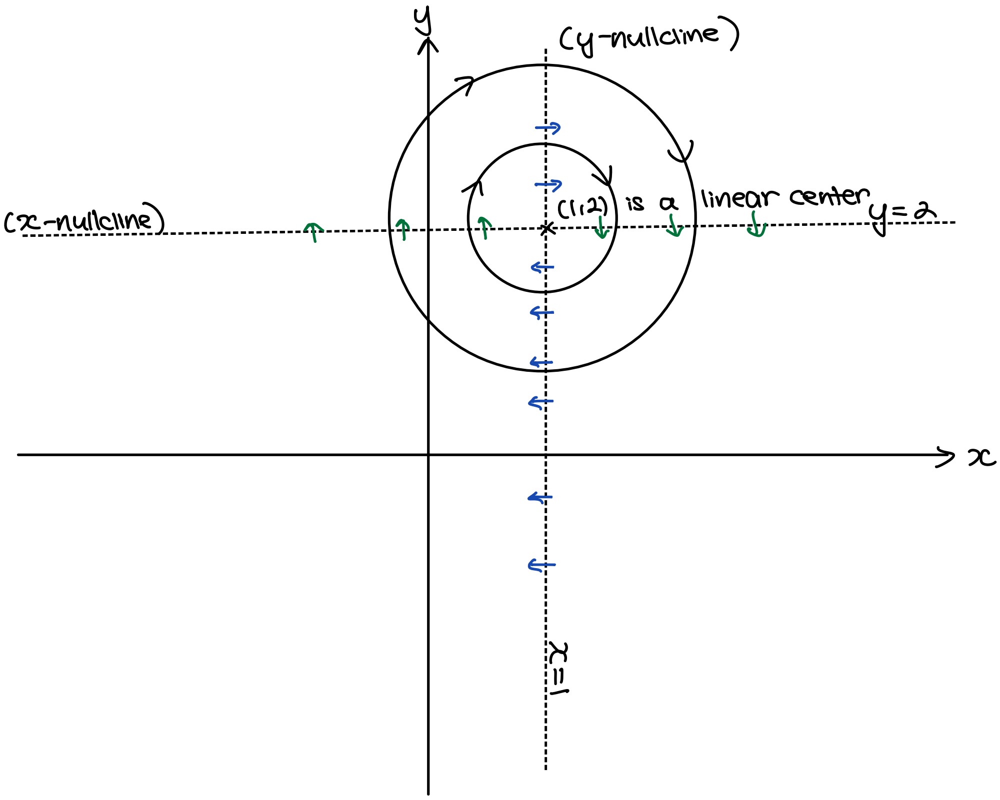
and hence

$$x = 1.$$

(ii) A fixed point must lie on both the $x-$ and $y-$ nullclines. Hence, the intersection of the line $x = 1$ with $y = 2$ is the point $(1, 2)$, which must correspond to the only fixed point in the above system.

(iii) See diagram on the next page.

(iii) $\begin{cases} \dot{x} = y - 2 \\ \dot{y} = 1 - x \end{cases}$



① Along y-nullcline (at $x=1$), $\dot{y}=0$ — Vector field pointing

$$\therefore \dot{x} = y - 2 = \begin{cases} >0 & \text{if } y>2 \\ <0 & \text{if } y<2 \end{cases} \Rightarrow \begin{matrix} \rightarrow \\ \leftarrow \end{matrix}$$

② Along x-nullcline (at $y=0$), $\dot{x}=0$ — Vector field pointing

$$\therefore \dot{y} = 1 - x = \begin{cases} <0 & \text{if } x>1 \\ >0 & \text{if } x<1 \end{cases} \quad \begin{matrix} \downarrow \\ \uparrow \end{matrix}$$

Example 29. Consider the following system of ODEs:

$$\begin{cases} \dot{x} = x - y, \\ \dot{y} = x^2 - 4. \end{cases}$$

(i) Find all fixed points.

(ii) Find the linearization of each fixed point(s).

(iii) Compute the eigenvalues of the Jacobian at each fixed point(s) and use these to classify any hyperbolic fixed point(s).

(iv) For every hyperbolic fixed point(s) with no imaginary parts, find an eigenvector for each eigenvalue found in (iii).

(v) Find all $x-$ and $y-$ nullcline(s) to the above system.

(vi) Sketch a phase portrait for the above system. Your phase portrait should demonstrate the information you have computed in parts (i) to (v).

Suggested Solutions:

(i) The fixed points are obtained by solving the following system:

$$\begin{cases} x - y &= 0 \\ x^2 - 4 &= 0 \end{cases}$$

This yields $x = \pm 2, y = \pm 2$; ie the two fixed points are $(-2,-2)$ and $(2,2)$.

(ii) Since $\mathbf{f}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} x - y \\ x^2 - 4 \end{bmatrix}$, we have

$$\nabla \mathbf{f}(x,y) = \begin{bmatrix} 1 & -1 \\ 2x & 0 \end{bmatrix}.$$

This implies that

$$\nabla \mathbf{f}(-2,-2) = \begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix}, \quad \nabla \mathbf{f}(2,2) = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}.$$

In other words, we have (for $\boldsymbol{\eta}(t) = \mathbf{x}(t) - \mathbf{x}^*$ with appropriate fixed points as from (i)), the linearization at $(-2,-2)$ is given by

$$\dot{\boldsymbol{\eta}} = \begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix} \boldsymbol{\eta}.$$

Similarly, the linearization at $(2,2)$ is given by

$$\dot{\boldsymbol{\eta}} = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix} \boldsymbol{\eta}.$$

(iii) At $(-2,-2)$, the eigenvalues are given by $\frac{1}{2}(1 \pm \sqrt{17})$, where $\frac{1}{2}(1+\sqrt{17}) > 0$ and $\frac{1}{2}(1-\sqrt{17}) < 0$. Hence, $(-2,-2)$ is a hyperbolic fixed point that corresponds to a saddle-node.

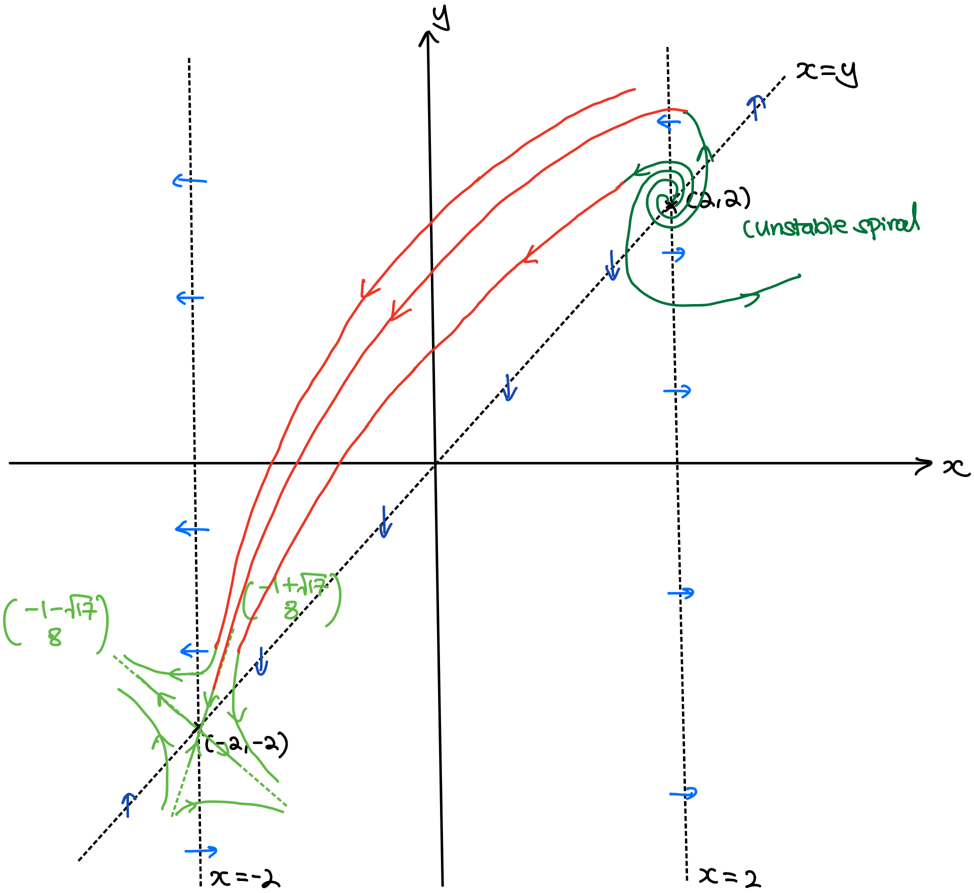
At $(2,2)$, the eigenvalues are given by $\frac{1}{2}(1 \pm \sqrt{17})$, where $\frac{1}{2}(1 \pm \mathrm{i}\sqrt{15})$. The real part of each of these eigenvalues is $\frac{1}{2} > 0$. Hence, $(2,2)$ is a hyperbolic fixed point that corresponds to an unstable spiral.

(iv) By the requirement of the question, it suffices to find the corresponding eigenvectors at $(-2,-2)$. For $\lambda = \frac{1}{2}(1+\sqrt{17})$, the corresponding eigenvector is given by $\begin{bmatrix} -1-\sqrt{17} \\ 8 \end{bmatrix}$. Similarly, for $\lambda = \frac{1}{2}(1-\sqrt{17})$, the corresponding eigenvector is given by $\begin{bmatrix} -1+\sqrt{17} \\ 8 \end{bmatrix}$.

(v) The $x-$nullclines are obtained by solving $\dot{x} = 0$. This corresponds to the line $x = y$.
The $y-$nullclines are obtained by solving $\dot{y} = 0$. This corresponds to the lines $x = 2$ and $x = -2$.

(vi) See diagram on the next page.

(vi)



① Sketch the nullclines and vector fields along each nullclines

x-nullcline, $\dot{x}=0$, $x=y$. $\therefore \dot{y}=x^2-4=\begin{cases} >0 & \text{if } |x|>2 \\ <0 & \text{if } -2<x<2 \end{cases}$

y-nullcline, $\dot{y}=0$, $x=\pm 2$, $\therefore \dot{x}=x-y=\begin{cases} >0 & \text{if } y<x \\ <0 & \text{if } y>x \end{cases}$

② By Hartman-Grobman theorem, sketch local phase portraits around each fixed points.

③ Connect the vector fields accordingly.

Lyapunov Stability.
Some Definitions:

- The *open ball* of radius $r > 0$ about $\mathbf{x}^*$ is defined as
$$B(\mathbf{x}^*, r) = \{\mathbf{x} \in \mathbb{R}^2 : ||\mathbf{x} - \mathbf{x}^*|| < r\}.$$
- We say that $\mathbf{x}^*$ is *attracting* if there exists $r > 0$ so that given any $\mathbf{x}(0) \in B(\mathbf{x}^*, r)$, we have
$$\mathbf{x}(t) \to \mathbf{x}^*$$
as $t \to \infty$.
(Intuitively, we say that the point is attracting if we can find "a region" containing $\mathbf{x}^*$ such that if we start from any point in the region, then we have $\mathbf{x} \to \mathbf{x}^*$.)
- We say that $\mathbf{x}^*$ is *globally attracting* if we can replace $B(\mathbf{x}^*, r)$ by $\mathbb{R}^2$.
(Intuitively, this means that starting from any point in $\mathbb{R}^2$, it will eventually tend to $\mathbf{x}^*$.)
- We say that $\mathbf{x}^*$ is *Lyapunov stable* if for any given $\varepsilon > 0$, there exists $r > 0$ so that for every $\mathbf{x}(0) \in B(\mathbf{x}^*, r)$, we have
$$\mathbf{x}(t) \in B(\mathbf{x}^*, \varepsilon)$$
for all $t \geq 0$.
(Intuitively, if we pick any starting point in some region containing $\mathbf{x}^*$, the trajectories will be bounded in some other region containing $\mathbf{x}^*$.)
- A fixed point that is Lyapunov stable but not attracting is *neutrally stable*.
- A fixed point that is Lyapunov stable and attracting is *asymptotically stable*.
- A fixed point that is neither attracting nor Lyapunov stable is *unstable*.

The classification of the fixed point $(0, 0)$ corresponding to all three real canonical forms is the content of Homework 7 Problem 1. Thus, instead of providing the solution to this problem, I shall classify the Lyapunov stability of some of the common types of fixed points.

Types of Fixed Point	Lyapunov Stability	Attracting	Stability
Stable Node	✓	✓	Asymptotically Stable
Unstable Node	×	×	Unstable
Saddle Node	×	×	Unstable
Stable Degenerate Node	✓	✓	Asymptotically Stable
Unstable Degenerate Node	×	×	Unstable
Center	✓	×	Neutrally Stable
Stable Spiral	✓	✓	Asymptotically Stable
Unstable Spiral	×	×	Unstable

9 Discussion 9

Potential and Hamiltonian Flows.

A **gradient flow/system** is a system that can be expressed as

$$\dot{\mathbf{x}} = -\nabla V(\mathbf{x}),$$

where $V : \mathbb{R}^2 \to \mathbb{R}$. We sometimes call V the potential/potential energy function of the gradient flow. (Compare this with the one-dimensional case, where we have $\dot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}$.)

Explicitly, we have

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \begin{cases} \dot{x} = -\frac{\partial V}{\partial x} \\ \dot{y} = -\frac{\partial V}{\partial y}. \end{cases}$$

A **Hamiltonian flow/system** is a system that can be expressed as

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla H.$$

Explicitly,

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y} \\ \dot{y} = -\frac{\partial H}{\partial x} \end{cases}$$

for some smooth function H (known as the Hamiltonian of the system).
Due to physical reasons, we often replace y by p to represent momentum (as a proxy for velocity, given by $\dot{x}$).

The following table summarizes the difference between these flows.

Properties	Potential Flow	Hamiltonian Flow
Governing Equations	$\begin{cases} \dot{x} = -\frac{\partial V}{\partial x} = f_1 \\ \dot{y} = -\frac{\partial V}{\partial y} = f_2 \end{cases}$	$\begin{cases} \dot{x} = \frac{\partial H}{\partial y} = f_1 \\ \dot{y} = -\frac{\partial H}{\partial x} = f_2 \end{cases}$
Is of such form if	$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$	$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 0$
Obtain V or H by solving	$\begin{cases} \frac{\partial V}{\partial x} = -f_1 \\ \frac{\partial V}{\partial y} = -f_2 \end{cases}$	$\begin{cases} \frac{\partial H}{\partial y} = f_1 \\ \frac{\partial H}{\partial x} = -f_2 \end{cases}$
$\frac{\mathrm{d}}{\mathrm{d}t}$ for all t	$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t), y(t)) < 0$; Equality at fixed point (x^*, y^*)	$\frac{\mathrm{d}}{\mathrm{d}t} H(x(t), y(t)) = 0$
Critical points of V or H are	Fixed points	Fixed points
Isolated local minima are Isolated local maxima are Isolated local saddles are	(Locally) Asymptotically Stable (Locally) Unstable (Locally) Saddle nodes	(Locally) Nonlinear centers (Locally) Nonlinear centers (Locally) Saddle nodes
Level curves of V or H are	Orthogonal to direction of flow	Parallel/Along the direction of flow
V or H are examples of	**Lyapunov Functions**	**Conserved Quantities**

Note that at critical points (ie fixed points) of V or H (which we denote by G for convenience), to determine whether it is a local minima/maxima/saddle, we look at the **Hessian**, given by

$$\nabla^2 G(x, y) = \begin{bmatrix} \frac{\partial^2 G}{\partial x^2} & \frac{\partial^2 G}{\partial x \partial y} \\ \frac{\partial^2 G}{\partial x \partial y} & \frac{\partial^2 G}{\partial y^2} \end{bmatrix} (x, y).$$

Let $\lambda_1 \le \lambda_2$ be the eigenvalues of $\nabla^2 G(x^*, y^*)$ (ie evaluated at the critical points).

- If $\lambda_2 \ge \lambda_1 > 0$, then (x^*, y^*) is a local minimum.
- If $\lambda_1 \le \lambda_2 < 0$, then (x^*, y^*) is a local maximum.
- If $\lambda_1 < 0 < \lambda_2$, then (x^*, y^*) is a (local) saddle point.

A related concept is to consider the time taken by a trajectory in a Hamiltonian system. Suppose that we are looking at the time taken T to go from $x = a$ to $x = b$. This is then given by

$$T = \int_a^b \mathrm{d}t(x) = \int_a^b \frac{\mathrm{d}t}{\mathrm{d}x}\mathrm{d}x = \int_a^b \frac{1}{\dot{x}}\mathrm{d}x = \int_a^b \frac{1}{p}\mathrm{d}x.$$

Using the fact that a trajectory remains on level curves of the Hamiltonian, we can determine p as a function of x and perform the relevant integration as required.

Lyapunov Functions and Conserved Quantities

We say that the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is conservative if there exists a continuous function $E : \mathbb{R}^2 \to \mathbb{R}$ (known as a **conserved quantity**) such that:

- The function E is not constant on any open ball $B(\mathbf{x}_0, \delta)$, and
- $E(\mathbf{x}(t))$ is constant on each trajectory. Mathematically, if E is continuously differentiable (on top of being continuous), we have $\frac{\mathrm{d}}{\mathrm{d}t}E(\mathbf{x}(t)) = 0$ for all time t.

If we can find such a conserved quantity E, we then have that if the isolated fixed point $\mathbf{x}^*$ corresponds to a critical point for E, we thus have that

- The system cannot have an attracting fixed point, and
- If $\mathbf{x}^*$ is a local minimum or maximum for E, it then corresponds to a nonlinear center for the system.

Example: A Hamiltonian system is a conserved system, with the Hamiltonian as the conserved quantity. This was proven in the previous concept on Hamiltonian flow that $\frac{\mathrm{d}}{\mathrm{d}t}H(\mathbf{x}(t)) = 0$ for all time t.

We say that the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has a **Lyapunov function** $F(t)$ if there exists a continuously differentiable function $F : \mathbb{R}^2 \to \mathbb{R}$ and a $\delta > 0$ such that for all $\mathbf{x} \in B(\mathbf{x}_0, \delta)$, we have:

- $F(\mathbf{x}) \geq 0$ with equality if and only if $\mathbf{x} = \mathbf{x}^*$, and
- $\nabla F \cdot \mathbf{f}(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$. Mathematically, this is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\mathbf{x}(t)) = \nabla F \cdot \dot{\mathbf{x}} = \nabla F \cdot \mathbf{f}(\mathbf{x}) < 0$$

  for all $\mathbf{x} \neq \mathbf{x}^*$.

Then, we have the following properties for the system:

- If $\mathbf{x}^*$ is an isolated fixed point, then it is asymptotically stable at that point.
- There are no closed orbits in the neighborhood of $\mathbf{x}^*$.

Example: A potential system is one with a Lyapunov function. The choice of Lyapunov function here would be the potential V plus a constant C large/small enough such that $V \geq 0$ for all $\mathbf{x}$ and equality holds when $\mathbf{x} = \mathbf{x}^*$ (basically, the first condition). Thus, recalling that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = -\nabla V$ for a potential flow, we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}V(\mathbf{x}(t)) = \nabla V \cdot \mathbf{f}(\mathbf{x}) = -|\nabla V|^2 < 0$$

for all $\mathbf{x} \neq \mathbf{x}^*$. This implies that the second condition for a Lyapunov function holds.

Hence, this is consistent with the fact that all potential flows with an isolated fixed point will correspond to an asymptotically stable fixed point. Furthermore, from the second conclusion drawn from the existence of a Lyapunov function, we know that there are no closed orbits in the neighborhood of $\mathbf{x}^*$ for a potential flow. In fact, such a statement is true globally (ie there are no closed orbits for a potential flow) if $\frac{\mathrm{d}}{\mathrm{d}t}V(\mathbf{x}(t)) < 0$ is true globally except at the fixed point $\mathbf{x}^*$.

There is no general strategy that one can use to employ Lyapunov functions/conserved quantities except by guessing/practice. We shall look at an example of such below.

Example 30. Consider the system

$$\begin{cases} \dot{x} = -2x + \varepsilon x^2 \\ \dot{y} = -2y \end{cases} \tag{8}$$

for some fixed constant $\varepsilon \geq 0$.

(i) Show that the above system is a potential flow.

(ii) Determine a function $V(x, y)$ corresponding to the potential of the gradient flow.

(iii) By computing $\nabla^2 V(x, y)$, determine the stability of the fixed point $(0, 0)$.

(iv) By sketching level curves of the potential function, sketch a phase portrait for the system above when $\varepsilon = 0$.

(i) Since $f_1 = -2x + \varepsilon x^2$ and $f_2 = -2y$, we observe that

$$\frac{\partial f_1}{\partial y} = 0 = \frac{\partial f_2}{\partial x}.$$

Hence, the above system is a potential flow.

(ii) To determine a corresponding potential function, we attempt to solve the following system

$$\begin{cases} \frac{\partial V}{\partial x} = 2x - \varepsilon x^2 \\ \frac{\partial V}{\partial y} = 2y. \end{cases}$$

Integrating the first equation, we have

$$V(x, y) = x^2 - \frac{\varepsilon}{3} x^3 + f(y)$$

for some arbitrary function f depending only on y. Similarly, by integrating the second equation, we have

$$V(x, y) = y^2 + g(x)$$

for some arbitrary function g depending only on x. By comparing the two different functional forms of V, we then deduce that

$$V(x, y) = x^2 - \frac{\varepsilon}{3} x^3 + y^2 + C.$$

(That is, $f(y) = y^2 + C$ and $g(x) = x^2 - \frac{\varepsilon}{3} x^3 + C$ necessarily.) Here, C is an arbitrary constant that we are free to choose, and for most cases, it suffices to pick $C = 0$.

(iii) One can easily check that $(0, 0)$ is a fixed point. Hence, we compute $\nabla^2 V(x, y)$, and obtain

$$\nabla^2 V(x, y) = \begin{bmatrix} 2 - 2\varepsilon x & 0 \\ 0 & 2 \end{bmatrix}.$$

This implies that

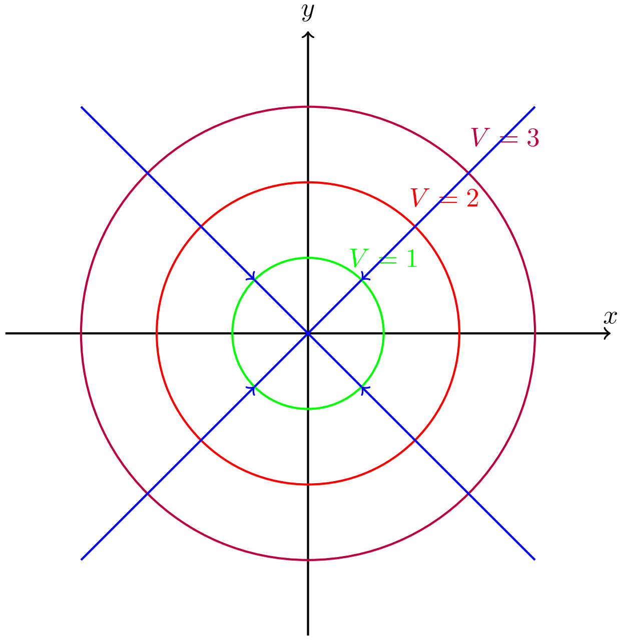
$$\nabla^2 V(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

The eigenvalues are both positive (ie 2, since this is a diagonal matrix so the diagonal entries are the eigenvalues). This implies that $(0, 0)$ is an **stable** fixed point (see the table above in the first page of this supplement). Here, we remark that the above analysis is equivalent to the linearization approach that we did in the last two weeks.

(iv) When $\varepsilon = 0$, we have that $V(x, y) = x^2 + y^2$. We will attempt to sketch the curves for $V = 1$ (that is $x^2 + y^2 = 1$, a circle centered at the origin with radius 1), $V = 2$, etc. Using the fact that

- $\frac{\mathrm{d}}{\mathrm{d}t} V \leq 0$ for all t with equality holds only at the fixed point, and
- Flows in a potential system are orthogonal to the level curves,

we must sketch solution curves to the orthogonal to level curves in the direction of decreasing V. Synthesizing the above information, we thus have



Example 31. Consider the following second-order differential equation

$$\begin{cases} \ddot{x} + x = 0, \\ x(0) = 0, \\ \dot{x}(0) = 1. \end{cases}$$

(i) Set $p = \dot{x}$. Convert the above differential equation into a linear system with $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$.

(ii) Show that the system in (i) is Hamiltonian and compute a corresponding Hamiltonian.

(iii) Compute the time T in which $x(t)$ first equal 1.

(iv) By sketching level curves of the Hamiltonian, sketch a phase portrait for the system in (i).

Suggested Solution:

(i) The corresponding system is given by

$$\begin{cases} \dot{x} = p \\ \dot{p} = -x. \end{cases}$$

(ii) Observe that with $f_1 = p$ and $f_2 = -x$, we have

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial p} = 0$$

and hence the system is Hamiltonian. To determine a corresponding Hamiltonian, we attempt to solve

$$\begin{cases} \frac{\partial H}{\partial p} = p \\ \frac{\partial H}{\partial x} = x. \end{cases}$$

Integrating the first equation, we have

$$H(x,p) = \frac{p^2}{2} + f(x).$$

where f is some arbitrary function depending only on x. Similarly, integrating the second equation yields

$$H(x,p) = \frac{x^2}{2} + g(p).$$

By comparing the two different functional forms of H, we then deduce that

$$H(x,p) = \frac{x^2}{2} + \frac{p^2}{2}.$$

(That is, $f(x) = \frac{x^2}{2}$ and $g(p) = \frac{p^2}{2}$ necessarily.)

(iii) As the system starts out from $H = \frac{1}{2}x(0)^2 + \frac{1}{2}(\dot{x}(0))^2 = \frac{1}{2}$, we must have that this value stays constant throughout this (unique) trajectory. Thus, along the trajectory from $x = 0$ to $x = 1$, we have

$$H = \frac{1}{2} = \frac{1}{2}x^2 + \frac{1}{2}p^2.$$

Thus,

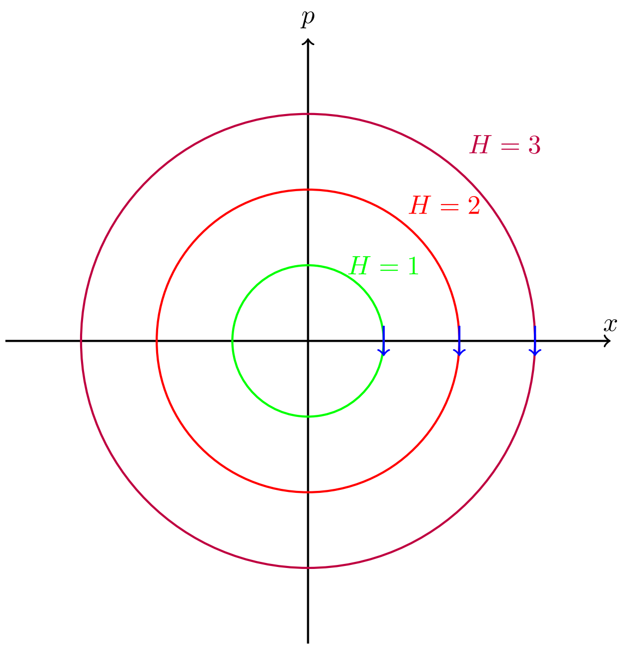
$$p = \sqrt{1 - x^2}.$$

Here, we have picked the positive square root as x is increasing till $\dot{x} = 0$ and $\dot{x}$ must be non-negative till $x = 1$ when $p = \dot{x}$ first attains 0. Thus, from the formula above, we have

$$T = \int_0^1 \frac{1}{\dot{x}} \mathrm{d}x = \int_0^1 \frac{1}{p} \mathrm{d}x = \int_0^1 \frac{1}{\sqrt{1 - x^2}} \mathrm{d}x = \frac{\pi}{2}.$$

(iv) You would just have to consider sketching different level curves (energy surfaces) for a given fixed value of H. Here, if $H = \frac{1}{2}$, then we have $x^2 + y^2 = 1$ corresponding to a circle of radius 1. Similarly, for $H = 1$, we have $x^2 + y^2 = 2$, corresponding to a circle of radius $\sqrt{2}$. From the above points, we know that the trajectories must lie on level curves (as the value of H does not change with time). Thus, the trajectories must be these circles centered at the origin with a radius corresponding to a given choice of H.

As we have already figured out the level curves for any given value of H. Writing down the system in such a form will only aid us to determine the direction of the flow along these circles. In fact, for $H = 1$, we consider the point $(x, p) = (\sqrt{2}, 0)$ on the circle and note that $(\dot{x}, \dot{p})|_{(\sqrt{2},0)} = (0, -\sqrt{2})$. This implies that the arrow must point in the negative p direction. A similar calculation for arbitrary H reveals that the trajectories must go in circles, in the clockwise direction. Synthesizing the above information, we then have:



Example 32. For the following system:

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y^3, \end{cases}$$

find a Lyapunov function for the fixed point $(0,0)$ of the system.

Suggested Solutions:

Consider the function

$$F(x,y) = x^2 + y^2.$$

Indeed, we have that $F \geq 0$ with equality if and only if $(x,y) = (0,0)$. Furthermore, one can check that

$$\frac{\mathrm{d}}{\mathrm{d}t} F = 2x\dot{x} + 2y\dot{y} = -2x^2 - 2y^4 < 0$$

for all $(x,y) \neq (0,0)$. Thus, this is a Lyapunov function for the fixed point $(0,0)$, implying that the fixed point must be asymptotically stable.

Example 33. Consider the following autonomous differential equation system:

$$\begin{cases} \dot{x} = -x + y^2 \\ \dot{y} = y - x^2. \end{cases}$$

Identify the fixed points of this equation, and show (either by linearizing the equation or by some other method) whether they are stable or not.

Suggested Solutions:

We first obtain the fixed points by solving $f_1 = -x + y^2 = 0$ and $f_2 = y - x^2 = 0$. The second equation yields

$$y = x^2$$

and substituting this to the first yields

$$-x + (x^2)^2 = x(x^3 - 1) = 0.$$

We thus have two roots, $x = 0$ ad $x = 1$. Using $y = x^2$, we have $y = 0$ and $y = 1$ respectively. This implies that we have two fixed points, $(0,0)$ and $(1,1)$.

Note that

$$\nabla \mathbf{f} = \begin{bmatrix} -1 & 2y \\ -2x & 1 \end{bmatrix}.$$

Thus, for $(0,0)$, we have

$$\nabla \mathbf{f}(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

with eigenvalues -1 and 1 implying that $(0,0)$ is a saddle point (since this corresponds to a hyperbolic fixed point and such an analysis for the nonlinear fixed point is true due to Hartman-Grobman Theorem), which is thus unstale.

For $(1,1)$, we have

$$\nabla \mathbf{f}(1,1) = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}.$$

The eigenvalues are given by $\pm\sqrt{3}\mathrm{i}$. Now, we claim that this is a Hamiltonian flow. This is done by considering

$$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = -1 + 1 = 0.$$

To obtain the Hamiltonian, one can consider solving

$$\begin{cases} \frac{\partial H}{\partial y} = -x + y^2 \\ \frac{\partial H}{\partial x} = -y + x^2 \end{cases}.$$

We can apply the usual techniques to solve for H. Here, H is given by

$$H(x,y) = x^3 + y^3 - xy.$$

Next, we would like to show that $(1,1)$ is a stable point. Consider

$$\nabla^2 H(x,y) = \begin{bmatrix} 6x & -1 \\ -1 & 6y \end{bmatrix},$$

which we then have

$$\nabla^2 H(1,1) = \begin{bmatrix} 6 & -1 \\ -1 & 6 \end{bmatrix},$$

with eigenvalues given by 5 and 7 which are both positive. This implies that the fixed point $(1,1)$ corresponds to a local minimum. Hence, we have that the fixed point $(1,1)$ is actually a nonlinear center, and is thus stable.

10 Discussion 10

Index Theory.

Definition:

- Let C be a smooth, simple curve oriented counter-clockwise that does not intersect any fixed points of the system
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$
where $\mathbf{f}$ is continuously differentiable. Explicitly,
$$\begin{cases} \dot{x_1} &= f_1(x(t), y(t)) \\ \dot{x_2} &= f_2(x(t), y(t)). \end{cases}$$
The **index** of the closed curve C with respect to the vector field $\mathbf{f}$ is given by
$$I_C = \frac{1}{2\pi} \oint_C \mathrm{d}\phi$$
where
$$\phi = \tan^{-1}\left(\frac{f_1}{f_2}\right).$$
- If the curve C is parameterized by $\mathbf{r}(t) = (x(t), y(t))$ with $0 \le t < 2\pi$, then we have
$$I_c = \frac{1}{2\pi}\int_0^{2\pi} \frac{f_1\nabla f_2 - f_2 \nabla f_1}{f_1^2 + f_2^2} \cdot \mathrm{r}'(t)\mathrm{d}t.$$

Properties: From the expression obtained above, if C is parameterized, the idea is that a fixed point corresponds to $f_1 = f_2 = 0$ and thus, a singularity in $\frac{f_1\nabla f_2 - f_2\nabla f_1}{f_1^2+f_2^2}$, and hence we would expect that the index of the curve circling the fixed point would not necessarily be 0. (Define $\mathbf{F} = \frac{f_1\nabla f_2 - f_2\nabla f_1}{f_1^2+f_2^2}$; then one can check that for $f_1^2 + f_2^2 > 0$ ie no fixed points in the domain, we have $\nabla \cdot \mathbf{F} = 0$.)

The following properties are thus inspired by such an idea and multi-variable calculus as follows:

- From a given phase portrait, the index can be computed by looking at the number of counter-clockwise revolutions made by the vector field as we complete one revolution in the counter-clockwise direction along the curve C.
- If C does not enclose any fixed points, then $I_C = 0$
(Consequence of Green's Theorem and $\nabla \cdot \mathbf{F} = 0$; the converse does not hold - see Example 36 below.)
- If we reverse all the arrows in the vector fields by changing $t \to -t$, then the index remains unchanged.
- Let C and C' be smooth simple closed curves. If C can be continuously deformed into C' without passing through a fixed point of the vector field, then
$$I_C = I_C'.$$
- If C is a closed orbit (regardless of orientation, ie clockwise/counter-clockwise), then $I_C = 1$.
(Draw a parallel with the Cauchy Integral formula if you have taken Math 132.)
- If C is a closed orbit, there must be at least one fixed point inside the trajectory. (If not, the absence of a fixed point inside the trajectory implies that $I_C = 0$, contradicting that closed orbits have $I_C = 1$.)
- The index of an isolated fixed point $\mathbf{x}^*$ is the index of any simple, smooth, closed curve C that encloses only $\mathbf{x}^*$. Thus,
$$I_{\mathbf{x}^*} = I_C.$$

- If a smooth, simple, closed curve C encloses fixed points $\mathbf{x}_1^*, ...\mathbf{x}_n^*$ with indices $I_1, ..., I_n$, then

$$I_C = I_1 + ... + I_n.$$

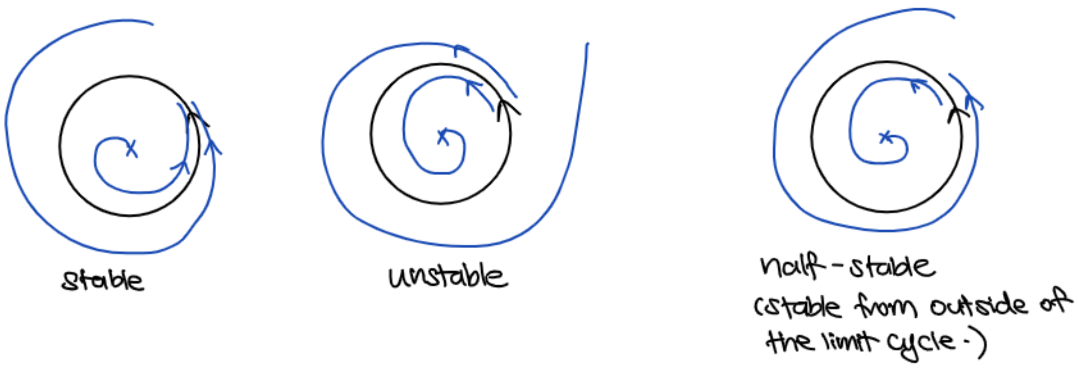
  (Draw a parallel with the Residue theorem if you have taken Math 132.)

- Thus, any closed orbit will enclose fixed points whose indices sum to 1.
- Nodes, spirals, degenerate nodes, and centers have an index 1, while saddle points have an index -1.

Limit Cycles and Periodic Solutions.

Definitions/Properties of Limit Cycles:

- A limit cycle is an isolated closed trajectory.
  Example: A center is not a limit cycle as the closed trajectory is not isolated (neighboring vector fields are also closed).
- Limit cycles can be stable/attracting, unstable or half-stable. Such classifications are usually done in polar coordinates.
- Gradient flows do not have closed trajectories, and thus, do not have a limit cycle.
- If a system has a Lyapunov function(s) corresponding to the relevant fixed points, then each fixed point is asymptotically stable. This implies that we do not have closed trajectories, let alone limit cycles.
- A closed orbit is equivalent to a periodic solution (if we allow ourselves to define constant solutions to be closed orbits).



Theorem 34. (Dulac's Criterion.) Let $\mathbf{f}(\mathbf{x})$ be a continuously differentiable vector field on a simply connected set $R \subseteq \mathbb{R}^2$. (Here, simply connected is with respect to the vector field $\mathbf{f}$, such that we have no singularities in the region R.) If there exists a continuously differentiable, real-valued function $g(\mathbf{x})$ so that

$$\nabla \cdot (g\mathbf{f}(\mathbf{x}))$$

has a sign throughout R, then there are no closed orbits of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in R.

Theorem 35. (Poincaré-Bendixson Theorem.) Suppose that

(i) $R \subseteq \mathbb{R}^2$ is closed and bounded.

(ii) $\mathbf{f}(\mathbf{x})$ is a continuously differentiable vector field on an open set containing R.

(iii) R does not contain any fixed points.

(iv) There exists a trajectory C of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ confined to R.

Then, R contains a closed orbit.

To apply the Poincaré-Bendixson Theorem, it is usually easy to show that (i) to (iii) holds. To show that (iv) holds, one will have to appeal to the construction of **trapping regions** (ie vector field pointing inwards towards a region that we construct (which is known as a **trapping region**)).

Bifurcations in 2D.

The basic idea is that bifurcations in 2D can be analyzed using a similar strategy as for the case in 1D. In other words, we look at how the number of fixed points (in 2D) changes as the bifurcation parameter μ changes, which usually involves the change in the number of roots to a quadratic equation (saddle-node), “colliding” roots (transcritical bifurcation), or pitchfork bifurcation of roots. **The only difference is that the stability of each fixed point as μ changes are analyzed in 2D (with the help of nullclines if needed).**

We shall look at an example of this below.

Example 36. Consider the following system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x(2-x). \end{cases}$$

(i) Show that $(0,0)$ is a nonlinear center.

(ii) Sketch the phase portrait of this system.

(iii) Consider a curve C of radius sufficiently small (at most 1) centered at the origin and oriented counter-clockwise. Use the phase portrait in (ii) to deduce that the value of the index $I_C = 1$.

(iv) Consider a curve C' of radius 3 centered at the origin and oriented counter-clockwise. Without using the phase portrait, compute the index $I_{C'}$.

Suggested Solution:

(i) One can check that $(0,0)$ is a fixed point of the system (along with $(2,0)$). Furthermore, it is a Hamiltonian system, since $\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x(2-x)) = 0$. One can then solve for the Hamiltonian by solving $\frac{\partial H}{\partial y} = y$ and $\frac{\partial H}{\partial x} = x(2-x)$ to obtain

$$H(x,y) = \frac{y^2}{2} + x^2 - \frac{x^3}{3}.$$

The Hessian is given by

$$\nabla^2 H(x,y) = \begin{bmatrix} 1-2x & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, at $(0,0)$, we have

$$\nabla^2 H(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and that both eigenvalues are positive, implying that $(0,0)$ is a local minimum of the Hamiltonian, and thus corresponds to a nonlinear center.

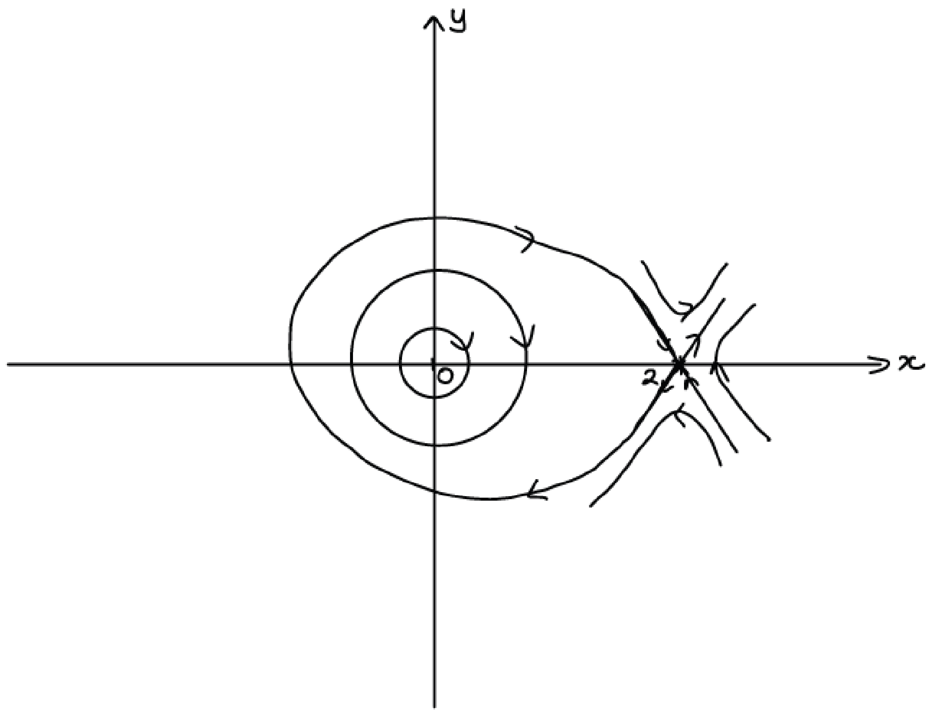
(ii) For the other fixed point, observe that

$$\nabla f = \begin{bmatrix} 0 & 1 \\ -2+2x & 0 \end{bmatrix}.$$

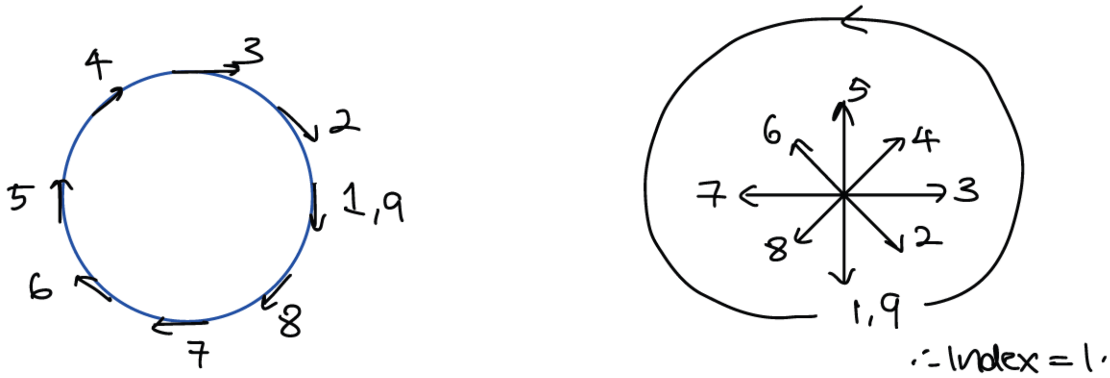
At $x = 2$, we have

$$\nabla f(2,0) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

has eigenvalues $-\sqrt{2}, \sqrt{2}$ with eigenvectors $\begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$, and hence corresponds to a saddle point. Thus, we have



(iii)



(iv) Note that C' encloses both fixed points. By the property of indices, as mentioned above, we have

$$I_{C'} = I_{(0,0)} + I_{(2,0)} = 1 - 1 = 0,$$

since we have that for a saddle, the index is -1, and that the index at $(0, 0)$ (which is a center) is 1 as seen in (iii).

Example 37. Consider the following ODE

$$\ddot{x} + \lambda \dot{x} + f(x) = 0$$

where x is twice continuously differentiable and f is smooth, with a constant $\lambda > 0$. Show that there are no periodic solutions other than a stationary fixed point $x(t) = c$ for all t for some constant c.
Hint: Use the weighting function $g(\mathbf{x}) = 1$ for all $\mathbf{x}$.

Suggested Solutions:

Let $y = \dot{x}$ and consider the following system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\lambda y - f(x). \end{cases}$$

The fixed points are obtained by solving $y = 0$ and $-\lambda y - f(x) = 0$. Since $y = 0$, we have

$$f(x) = 0$$

and thus x^* are solutions to the equation $f(x) = 0$. The fixed points are thus given by $(x, y) = (x^*, 0)$.

Note that it is challenging to construct a Lyapunov function. Furthermore, one can check that the system is not a potential flow. Thus, we have to turn to alternative tools. For this class, the only alternative tool that we have to rule out closed orbits (periodic solutions) is Dulac's criterion. By setting $g \equiv 1$, we compute (using $\mathbf{f} = (y, -\lambda y - f(x))^\intercal$):

$$\nabla \cdot (g\mathbf{f}) = \nabla \cdot \mathbf{f} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-\lambda y - f(x)) = -\lambda < 0.$$

Thus, the sign is maintained throughout $\mathbb{R}^2$. Furthermore, $\mathbf{f}$ is clearly continously differentiable on $\mathbb{R}^2$ and $\mathbb{R}^2$ is a simply connected set corresponding to $\mathbf{f}$. Furthermore, $g(\mathbf{x}) = 1$ is continuously differentiable. By Dulac's criterion, there are no closed orbits for the system. Thus, the only periodic solutions would be the fixed points.

Example 38. (Strogatz 7.1.5, Modified.) Consider the following system:

$$\begin{cases} \dot{x} = x - y - x(x^2+y^2), \\ \dot{y} = x + y - y(x^2+y^2). \end{cases}$$

In the usual polar coordinates (r, θ), we have

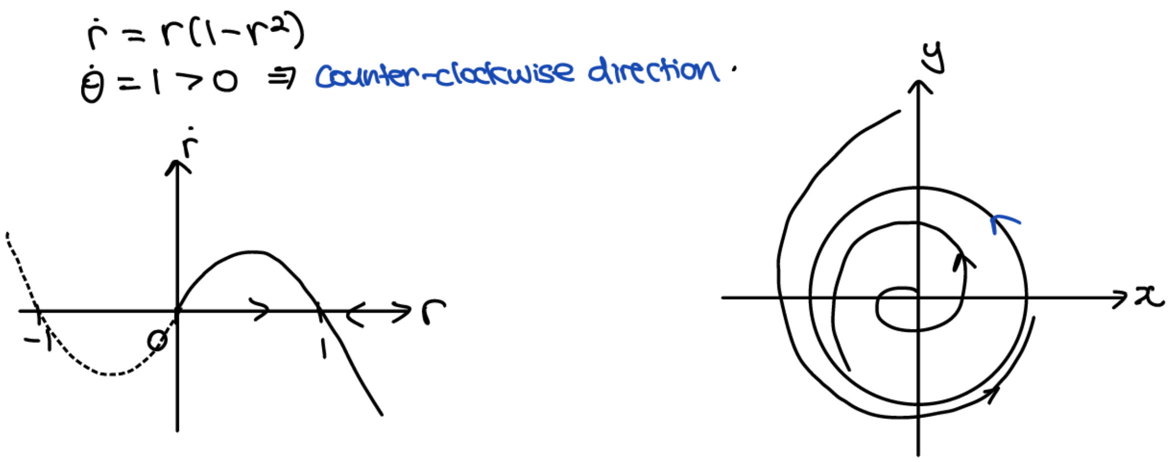
$$\begin{cases} \dot{r} = r(1-r^2), \\ \dot{\theta} = 1. \end{cases}$$

(i) Sketch the phase portrait for the above system.

(ii) Observe that $r = 1$ corresponds to a limit cycle. Classify the stability of this limit cycle.

(iii) Use the Poincaré-Bendixson theorem to prove that there exists a periodic solution.

(i)



(ii) It is a stable limit cycle.

(iii) First, we observe that the only fixed point is $(0,0)$. This is evident from the dynamics in polar coordinates, where we have fixed points if $\dot{r} = 0$ and $\dot{\theta} = 0$. For $\dot{\theta} = 0$, this is only possible if $r = 0$ (else it is well-defined and is equal to 1). Furthermore, we check that at $r = 0$, we indeed have $\dot{r} = 0$. Hence, $r = 0$ is the only fixed point that we have, which thus corresponds to the origin $(0,0)$.

Next, we construct a trapping region $R = \{(x,y) \in \mathbb{R}^2 : 1/2 \le \sqrt{x^2+y^2} \le 2\}$ (ie an annular region of radius $1/2$ and 2). Observe that

$$\dot{r}(r = 1/2) = \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^2\right) = \frac{3}{8} > 0,$$

and

$$\dot{r}(r = 2) = 2\left(1 - 2^2\right) = -6 < 0.$$

This implies that vector fields point radially outwards along the circle $r = 1/2$, and radially inwards along the circle $r = 2$. This implies that R is a trapping region.
Furthermore, R is a closed and bounded region in $\mathbb{R}^2$ that does not contain any fixed points. In addition, $\mathbf{f}(x,y) = \begin{bmatrix} x - y - x(x^2+y^2) \\ x + y - y(x^2+y^2) \end{bmatrix}$ is continuously differentiable in $\mathbb{R}^2$.

The conditions for Poincaré-Bendixson Theorem are satisfied, and thus R contains a closed orbit (periodic solution).

Example 39. Consider the following system:

$$\begin{cases} \dot{x} = \mu - x^2, \\ \dot{y} = x - y. \end{cases}$$

(i) Sketch the nullclines for different values of μ and for sufficiently small $|\mu|$.

(ii) Show that $\mu = 0$ is a bifurcation point and classify the type of bifurcation $\mu = 0$.

(iii) Sketch the phase portrait as a function of μ for sufficiently small $|\mu|$.

We shall attempt (i) and (ii) together as follows. The fixed points are given by solving $\mu = x^2$ and $x - y = 0$. This implies that $x = \pm\sqrt{\mu}$ and thus $y = x = \pm\sqrt{\mu}$. Henceforth, we have

- 2 fixed points if $\mu > 0$, at $(-\sqrt{\mu}, -\sqrt{\mu})$ and $(\sqrt{\mu}, \sqrt{\mu})$.
- 1 fixed point if $\mu = 0$, at $(0, 0)$, and
- 0 fixed points if $\mu < 0$.

Thus, $\mu = 0$ is a bifurcation point. Thus, we shall sketch nullclines for $\mu < 0$, $\mu = 0$, and $\mu > 0$ as follows: Furthermore, we can rigorously analyze this by linearization as follows. The Jacobian can be computed to be

$$\nabla \mathbf{f}(x, y) = \begin{bmatrix} -2x & 0 \\ 1 & -1 \end{bmatrix}.$$

For $\mu = 0$, we have

$$\nabla \mathbf{f}(0, 0) = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

The presence of a zero eigenvalue would render the Hartman-Grobman theorem ineffective. For $\mu > 0$, we have

$$\nabla \mathbf{f}(\sqrt{\mu}, \sqrt{\mu}) = \begin{bmatrix} -2\sqrt{\mu} & 0 \\ 1 & -1 \end{bmatrix}$$

with eigenvalues $-1, -2\sqrt{\mu}$ and eigenvectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 - 2\sqrt{\mu} \\ 1 \end{bmatrix}$; and

$$\nabla \mathbf{f}(-\sqrt{\mu}, -\sqrt{\mu}) = \begin{bmatrix} 2\sqrt{\mu} & 0 \\ 1 & -1 \end{bmatrix}$$

with eigenvalues $-1, 2\sqrt{\mu}$ and eigenvectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 + 2\sqrt{\mu} \\ 1 \end{bmatrix}$. Thus one can observe that the 0 eigenvalue branches out into a stable (and unstable) branch at $(\sqrt{\mu}, \sqrt{\mu})$ (and $(-\sqrt{\mu}, -\sqrt{\mu})$). This implies that the two fixed points at $\mu > 0$ correspond to one stable and one unstable point. In fact, we can see from the phase portraits/nullclines that $(0, 0)$ for $\mu = 0$ is a half-stable point. Hence, we can then conclude that $\mu = 0$ corresponds to a saddle-node bifurcation.

(i)

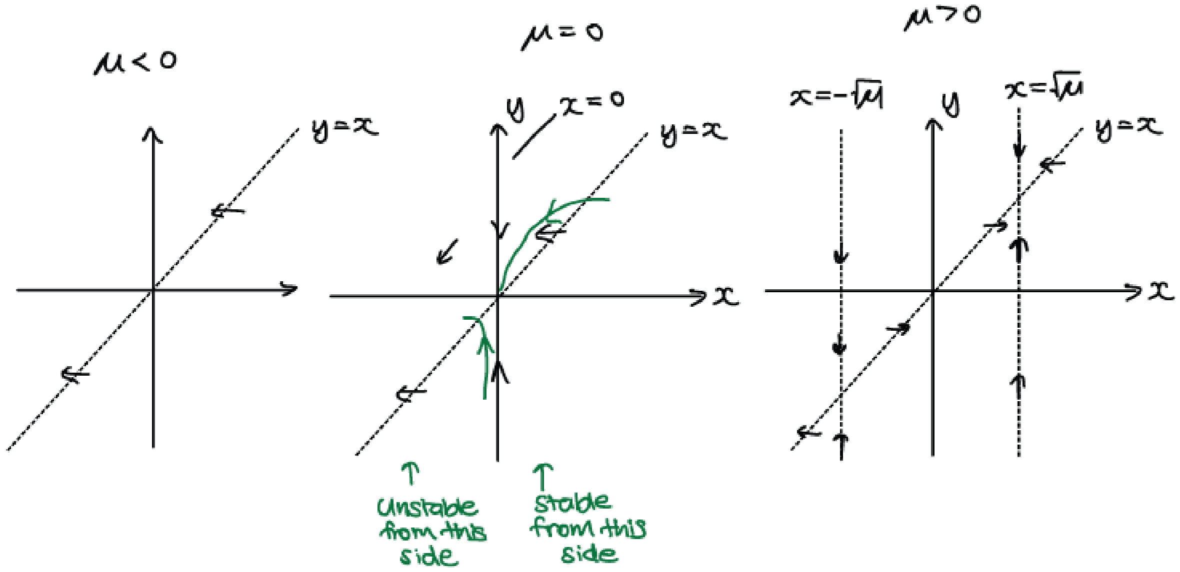
$$\begin{cases} \dot{x} = \mu - x^2 \\ \dot{y} = x - y \end{cases}$$

$\dot{x} = 0 \Rightarrow x = \pm\sqrt{\mu}$.

$$\dot{y} = x - y = \begin{cases} >0 & \text{if } y < x \\ <0 & \text{if } y > x \end{cases}$$

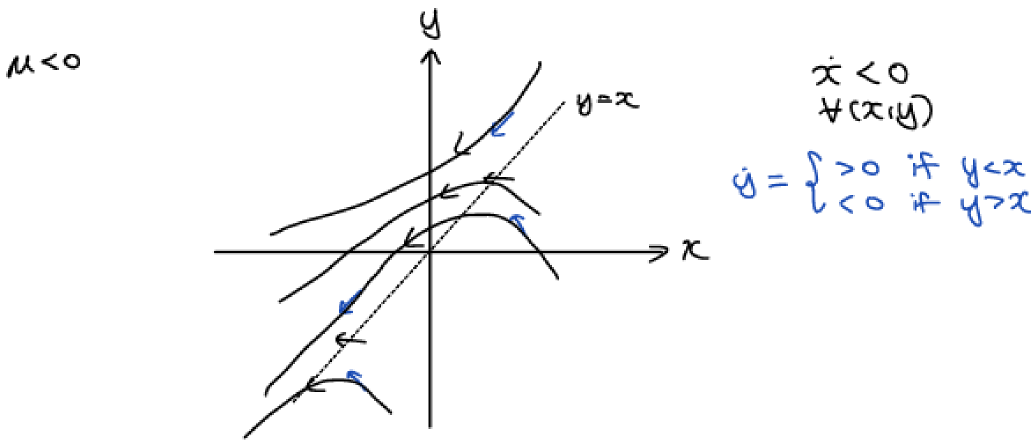
$\dot{y} = 0 \Rightarrow y = x$

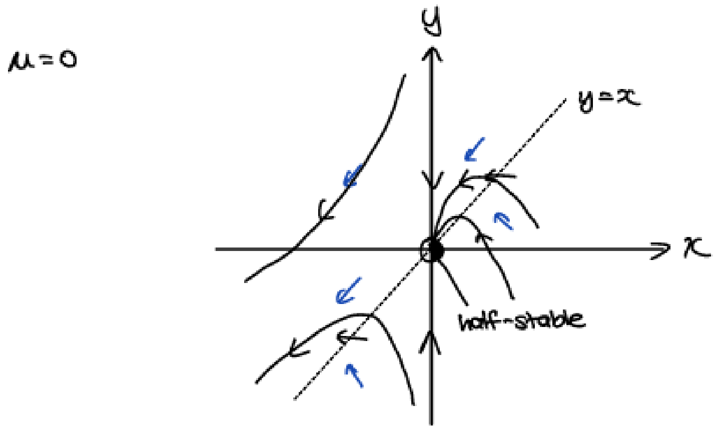
$$\dot{x} = \mu - x^2 = \begin{cases} >0 & \text{if } -\sqrt{\mu} < x < \sqrt{\mu} \\ <0 & \text{if } x > \sqrt{\mu} \text{ or } x < -\sqrt{\mu} \end{cases}$$

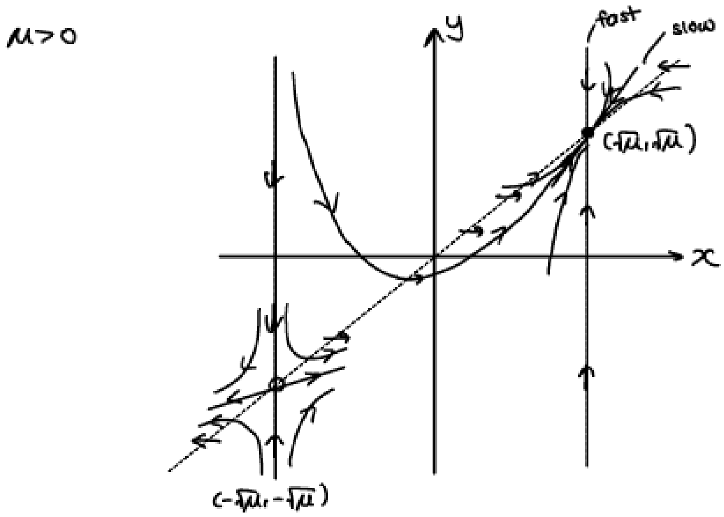


∴ (0,0) is "half-stable".

(iii)







References

[1] Steven H. Strogatz. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*. CRC Press, 2nd edition, 2015.