Math 174E - Discussion Supplements for Spring 24

Contents are motivated from [1] and lecture slides from Derek Levinson, which in turn were motivated from Moritz Voss. [1]

Contents

1 Discussion 1

Forward Contracts and Introduction to Options.

Key Terminologies/Definitions:

- A **financial asset** is an asset whose value is derived from a contractual right or ownership claim.
- A **financial security** is a tradable **financial asset**.[2]
- A **financial instrument** is a monetary contract between parties.
- A **financial derivative** is a financial instrument whose value (price) is derived from the value of other financial assets.
- An **ask price** is a market price at which a seller is asking for to sell one unit of an asset.
- A **bid price** is a market price at which a buyer is bidding for one unit of an asset.[3]
- A **forward contract** is an agreement between two parties to buy or sell an asset at a predetermined **forward price** (**delivery price**) at a predetermined future time (**maturity date**).
- A **spot contract** is an agreement between two parties to buy or sell an asset "on the spot" for its immediate market price (**spot price**).
- The party with a **long position** agrees to buy (and hence have more of) the underlying asset/derivative.
- The party with a **short position** agrees to (and hence have less of) the underlying asset/derivative.
- The **payoff** is the **market value** of a position.
  Example: Payoff of a forward contract = Value of contract at maturity.
  (i.e. How much do you, depending on your position, earn/get paid from it.)
- The **payoff function** describes how payoff varies with the spot price at maturity.
- An **option** is a contract that gives the holder the option/right to buy/sell an asset from/to the seller (writer) of an option at a predetermined price (**strike price**) **on/by** a predetermined future time (**maturity date**).
- A **call option** is an option that gives the holder the option (right) to ("call and ") buy an asset from the seller (writer) of the option (on/by the predetermined date).
- A **put option** is an option that gives the holder the option (right) to ("put and ") sell an asset from the seller (writer) of the option (on/by the predetermined date).
- A **premium/option price** is the price the buyer of an option pays (to get the premium benefit of having the option).
- **Writing** an option is the same as **selling** an option.
- An **American** (call/put) option is one that is **free** to be exercised **any time** before maturity.
- A **European** (call/put) option is one that **can only be** exercised **at** maturity.

Forward Contract:

- Forward price/unit, K.[4]
- Spot price/unit at maturity T, S_T.
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	Long Position ("Buyer")	Short Position ("Seller")
Payoff (function)/unit	$\underbrace{S_T - K}_{\text{Buy for } K\text{, value at } S_T}$	$\underbrace{K - S_T}_{\text{Sell for } K\text{, value at } S_T}$

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[2] This also commonly refers to any form of financial instruments.

[3] The **best** ask/bid price would be the lowest/highest ask/bid price for the asset.

[4] An alternative notation for this would be F, as seen in the textbook/lecture slides.

- Technical note: Recall that you can **always short** a financial asset **even though you don't have it**. The catch is that we are assuming that there is an infinite supply of the asset at the spot (best ask) price S_T that is always available. Hence, buying the required quantity of an asset to fulfil a contract is always possible on the maturity date, with the caveat that we should be looking at the **ask price**.

Relevant concepts on options will be discussed in one of the exercise problems below.

Exercise 1. (Exercise 1.1) Forward contracts on foreign exchange are very popular and can be used to *hedge* foreign currency risk. Suppose that, on May 21, 2020, the treasurer of a U.S. corporation knows that the corporation will pay £1,000,000 in 6 months (ie on November 21, 2020). In order to hedge against exchange rate, the treasurer decides to enter into a forward contract with an investment bank. The forward price is $F = \$1.2230$ per £ and the maturity date is November 21, 2020.

(a) What position in the forward contract should the treasurer take? What is the corresponding position of the investment bank in the forward contract? What are their commitments?

(b) What is the payoff (or market value) of the forward contract for the corporation on November 21, 2020, if the £/\$ exchange rate on that day turns out to be (i) \$1.3000 per £, or (ii) \$1.2000 per £? What is the market value of the forward contract for the investment bank in these two cases?

Suggested Solutions:

(a) Treasurer: **Long Position** (for the £). Investment Bank: **Short Position** (for the £).
Commitment: On November 21, 2020, the investment bank will give the U.S. corporation £1,000,000 in exchange for \$1.2230/£ × £1,000,000 = \$1,223,000 from the U.S. corporation.

Note that instead of exchanging the underlying asset (ie £), the long and short positions could equivalently exchange cash (USD). This is called **cash settlement**.

(b) Note: S_T represents the exchange rate at maturity, and K represents forward/delivery exchange rate.

(i) Treasurer = Long position: $S_T - K = 1.3000 - 1.2230 = +0.0770\ \$/£$.
Total payoff $= +0.0770$ \$/£ × £1,000,000 = $\boxed{+\$77{,}000}$.

Investment Bank = Short position: $K - S_T = 1.2230 - 1.3000 = -0.0770\ \$/£$.
Total payoff $= -0.0770$ \$/£ × £1,000,000 = $\boxed{-\$77{,}000}$.

(ii) Treasurer = Long position: $S_T - K = 1.2000 - 1.2230 = -0.2230\ \$/£$.
Total payoff $= -0.2230$ \$/£ × £1,000,000 = $\boxed{-\$22{,}300}$.

Investment Bank = Short position: $K - S_T = -(S_T - K) = 0.2230\ \$/£$.
Total payoff $= +0.2230$ \$/£ × £1,000,000 = $\boxed{+\$22{,}300}$.

Exercise 2. (Exercise 1.2) (Hull, Question 1.25.) A trader enters into a short forward contract on ¥100,000,000. The forward exchange rate is 0.0090 $/¥. How much does the trader gain or lose if the exchange rate at the end of the contract is

(a) 0.0084 $/¥.

(b) 0.0101 $/¥.

Suggested Solutions:

Forward Exchange Rate = K = 0.0090 $/¥.
Spot Exchange Rate = S_T = Given $/¥.
Position: Short.
Payoff $= \underbrace{K}_{\text{Forced to sell and get}} - \underbrace{S_T}_{\text{Value of currency}}$.

(a) Payoff Exchange Rate $= K - S_T = 0.0090 - 0.0084$ = + 0.0006 $/¥.
Total payoff = 0.0006 $/¥ × ¥100,000,000 = $\boxed{+\$60{,}000}$.

(b) Payoff Exchange Rate $= K - S_T = 0.0090 - 0.0101$ = − 0.0011 $/¥.
Total payoff = − 0.0011 $/¥ × ¥100,000,000 = $\boxed{-\$110{,}000}$.

Exercise 3. (Exercise 1.3) (Hull, Question 1.30.) On July 1, 2021, a company enters into a forward contract to buy ¥10,000,000 on January 1, 2022. On September 1, 2021, it enters into a forward contract to sell ¥10,000,000 on January 1, 2022. Describe the payoff from this strategy.

Suggested Solution:

Let X and Y be some positive real numbers.
Forward Contract 1, Long Position (for ¥): X \$/¥.
Forward Contract 2, Short Position (for ¥): Y \$/¥.

On Janurary 1, 2022, we are forced to pay \$10,000,000X to buy ¥10,000,000, to sell all that yen for a "revenue" of \$10,000,000Y.[5]

Total payoff = \$10,000,000Y – \$10,000,000X = \$10,000,000$(Y - X)$.

- If $Y > X$, total payoff is positive, and this is a good strategy.
- If $Y < X$, total payoff is negative, and this is not a good strategy.

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[5] All the "buying" and "selling" will happen on January 1, 2022. Hence, the payoffs are calculated with respect to the value of dollar on January 1, 2022 to avoid complicating the equation and introducing interest rates into it (this will be covered in a subsequent discussion).

Exercise 4. (Exercise 1.4) The following exercise explores the properties of a European put/call option. Recall that stock options are typical examples of financial derivatives (written on a stock).

(a) What is the crucial difference between a forward contract written on a stock and an option written on a stock?

(b) Would the holder of a European call option (put option) always exercise her right to buy the stock (sell the stock) at maturity T for the predetermined strike price K?

(c) Discuss the purpose of a European call and a European put option on a stock. Describe a situation where it would make sense for an investor to buy a call option or to buy a put option.

(d) Describe the payoff (a.k.a market value) at maturity T of following option positions as a function of the spot price of the underlying stock S_T at time T:

  (i) Long position in a call,
  (ii) Short position in a call,
  (iii) Long position in a put, and
  (iv) Short position in a put.

(e) The buyer of an option (long position) has to pay a **premium** to the seller/writer of the option (short position) at time $t = 0$ when both parties agree on the deal. Denote by $C_0(K, T)$ the price of a European call option at time $t = 0$ with strike price K and maturity T, and by $P_0(K, T)$ the price of a European put option at time $t = 0$ with strike price K and maturity T.

Describe the net profit (or loss) at maturity T of all option positions (i) to (iv) in (d) as a function of the price of the underlying stock S_T at time T.

Suggested Solutions:

(a) Forward Contract: Buyer/seller are **obliged/forced** to buy/sell the stock.
Option: The holder of the option has the **option** to buy/sell the stock (depending on if it is a call/put option).

(b) Without loss of generality, we will consider the European call option. Suppose that the spot price of the stock at maturity T is S_T.

- If $S_T > K$, the holder is allowed to buy stocks worth S_T only at a price of K. This would yield a profit of $S_T - K > 0$. Hence, the holder of the option **should** exercise the option.
- If $S_T \leq K$,[6] the holder is allowed to buy stocks worth S_T but at an exorbitant price of K. This would yield a payoff of $S_T - K \leq 0$. Hence, the holder of the option **should not** exercise the option.

(c) (Reference: Hull, Chapter 1.7 and 1.8). Let K be the strike price, S_T be the spot price at maturity. For our analysis, assume without loss of generality that the premium paid by the holder of the option is zero (this effect will be analyzed in (e)).

- **Hedging (Insurance).** If the investor wants to safeguard against increase in price of the stock (ie $S_T > K$), the investor can purchase a call option so that they are guaranteed to pay **at most** K for the stock (ie if $S_T > K$, they can choose not to exercise their option).

  On the other hand, if the investor wants to guarantee a minimum selling price of the stock to safeguard against decrease in the price of stock, the investor can purchase a put option so that they are guaranteed to sell the stock for **at least** K (ie if $S_T < K$, they can exercise their option to get the strike price K).
- **Speculating.** If the investor thinks that the stock price will go above the strike price, ie at maturity, $S_T > K$. Exercise a call option to buy the stock at price K will yield a profit of $S_T - K$.

  On the other hand, if the investor thinks that the stock price will go below the strike price i.e $S_T < K$, exercising a put option to sell the stock at price K will yield a profit of $K - S_T$.

Note: Arbitrage is yet another reason. For more information, refer to the lecture notes or the textbook.

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[6] When $S_T = K$, it doesn't matter. In fact, in practice, exact equality never happens if we look at enough decimal places!

(d) Let $(x)^+ = \min\{0, x\}$ be the positive part of x for any real number x.
(This implies that the $()^+$ function returns its argument if its positive, else it returns 0.)
Long position (L): Holder of the option.
Short position (S): The other party who sold the option.
Call = (C), Put = (P).

- (i) Payoff (L,C) $= \begin{cases} \underbrace{S_T - K}_{\text{Exercise call; pay } K \text{ to get } S_T.} & \text{if } S_T > K, \\ \underbrace{0}_{\text{Do not exercise; pay } K \text{ to get } S_T \text{ is not worth it.}} & \text{if } S_T \leq K, \end{cases} = (S_T - K)^+.$
- (ii) Payoff (S,C) $= \begin{cases} \underbrace{K - S_T = -(S_T - K)}_{\text{Holder exercised call since they would pay } K \text{ to get } S_T.} & \text{if } S_T > K, \\ \underbrace{0}_{\text{Holder did not exercise call since they would have paid } K \text{ to only get } S_T.} & \text{if } S_T \leq K, \end{cases} = -(S_T - K)^+.$
- (iii) Payoff (L,P) $= \begin{cases} \underbrace{0}_{\text{Do not exercise; sell for only } K \text{ when its worth } S_T.} & \text{if } S_T > K, \\ \underbrace{K - S_T}_{\text{Exercise put; sell for } K \text{ when its only worth } S_T} & \text{if } S_T \leq K, \end{cases} = (K - S_T)^+.$
- (iv) Payoff (S,P) = $-$ Payoff (L,P) = $-(K - S_T)^+$, since whatever the holder of the option profits from comes from the short position. (You can argue similarly for (ii).)

(e) Since the buyer has to pay a premium, this comes out of the profit in (d) in the long position. On the other hand, this is used to offset loss in (d) in the short position. The premium for the call and put options are $C_0(K, T)$ and $P_0(K, T)$ respectively from the problem.

- (i) Net profit (L,C): $(S_T - K)^+ - C_0(K, T)$,
- (ii) Net profit (S,C): $C_0(K, T) - (S_T - K)^+$,
- (iii) Net profit (L,P): $(K - S_T)^+ - P_0(K, T)$,
- (iv) Net profit (S,P): $P_0(K, T) - (K - S_T)^+$.

2 Discussion 2

(European) Options.

We start by recalling the following definitions:

- An **option** is a contract that gives the holder the option/right to buy/sell an asset from/to the seller (writer) of an option at a predetermined price (**strike price**) **on/by** a predetermined future time (**maturity date**).
- A **call option** is an option that gives the holder the option (right) to (“call and ”) buy an asset from the seller (writer) of the option (on/by the predetermined date).
- A **put option** is an option that gives the holder the option (right) to (“put and ”) sell an asset from the seller (writer) of the option (on/by the predetermined date).
- A **premium/option price** is the price the buyer of an option pays (to get the premium benefit of having the option).
- **Writing** an option is the same as **selling** an option.
- An **American** (call/put) option is one that is **free** to be exercised **any time** before maturity.
- A **European** (call/put) option is one that **can only be** exercised **at** maturity.
- The **payoff** is the **market value** of a position.
  Example: Payoff of an option = Value of option at maturity.
  (i.e. How much do you, depending on your position, earn/get paid from it.)
- The **payoff function** describes how payoff varies with the spot price at maturity.

We will supplement these with one more definition as follows:

- The **net profit** (net profit and loss) of a position in financial securities (assets, derivatives) is the difference between the payoff and the set-up cost.

In this class, we will be dealing solely with the European options.

Furthermore, some pointers for options are as follows:

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Payoff	Net Profit (Gain/Loss)
Does not include cost/revenue of option	**Includes** the cost/revenue of the option

- Taking stocks as an example of an asset associated with an option, we have

Price of an option/**Option price**	Price of an option **contract**
For 1 share of stock	(Usually) For 100 shares of stock[7]

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[7] This is more so in the US, and a different number will be specified if needed to be.

Exercise 5. (Exercise 2.1.) On May 21, 2020, the spot ask price of Apple stock was \$316.50, and the ask price of a call option with a strike price of \$320 and a maturity date of September is \$21.70. A trader is considering two alternatives:

(i) Buy 100 shares of the stock or

(ii) Buy 100 September call options (= 1 call option contract).

For each alternative, determine

(a) The upfront cost,

(b) The total gain if the stock price in September is \$400, and

(c) The total loss if the stock price in September is \$300.

Assume that the option is not exercised before September and positions are unwound (= closed out) at option maturity.

Suggested Solutions:

	(i) Buy 100 shares	**(ii) Buy 100 call options**
(a) Upfront cost	\$316.50 × 100 stocks = \$31,650.	\$21.70 × 100 options = \$2,170.
(b) Total gain, $S_T = \$400$	Value of Stocks: \$400/stock × 100 = \$40,000. Profit: $\$40{,}000 - \underbrace{\$31{,}650}_{\text{Cost of stocks}} = \boxed{+\$8{,}350}$.	$400 = S_T > K = 320$, exercise call. Payoff: $100(S_T - K) = \$8{,}000$. Profit: $\$8{,}000 - \underbrace{\$2{,}170}_{\text{Cost of Options}} = \boxed{+\$5{,}830}$.
(c) Total loss, $S_T = \$300$	Value of stocks: \$300/stock × 100 = \$30,000. Net Profit: $\$30{,}000 - \underbrace{\$31{,}650}_{\text{Cost of stocks}} = \boxed{-\$1{,}650}$.	$300 = S_T < K = 320$, **do not** exercise call. Payoff: 0. Net Profit: $\$0 - \underbrace{\$2{,}170}_{\text{Cost of Options}} = \boxed{-\$2{,}170}$.

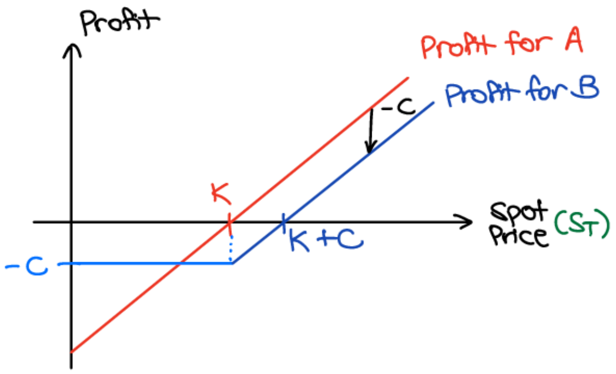
Moral of the story: Since options come with a premium, it might not be the best “option” for certain ranges of spot prices at maturity. Analysis should be done with respect to “what is the speculated range of the spot price at maturity” and “if a call option is worth it then”.

Exercise 6. (Exercise 2.2.) Trader A enters into a forward contract to buy an asset for \$1,000 in one year. Trader B buys a call option to buy the asset for \$1,000 in one year. The cost of the option is \$100. What is the difference between the positions of the traders? Show the profit as a function of the price of the asset in one year for the two traders.

Suggested Solutions:
Let K be the forward/strike price, S_T be the spot price at maturity, and $C(K,T)$ be the price of the call option.

- Net Profit for A = $\underbrace{S_T}_{\text{Value of asset}} - \underbrace{K}_{\text{Forward price; obliged to buy asset for this price.}}$.
- Net Profit for B = $\underbrace{(S_T - K)^+}_{\text{Payoff from call option}} - \underbrace{C(K,T)}_{\text{Call option premium}}$.

The following graphs shows the profit of each trader as a function of the spot price S_T in one year.



Profits

$A = S_T - K$

$B = (S_T - K)^+ - C$

$= \begin{cases} -C & \text{if } S_T < K, \\ S_T - K - C & \text{if } S_T > K. \end{cases}$

(which is just $A - C$.)

To match the scenario in the problem, set $K = \$1{,}000$ and $C(K,T) = \$100$.

Exercise 7. (Exercise 2.3.) In March, a U.S. investor instructs a broker to sell one July put option contract on a stock. The stock price is \$42, and the strike price is \$40. The option price is \$3. Explain what the investor has agreed to. Under what circumstances will the trade prove to to be profitable? What are the risks?

Suggested Solutions:

The U.S investor agreed on buying **100** stocks (assuming that an option contract represents 100 options) for \$40/stock at maturity (in July) if the other party chooses to exercise their put option.

(Short (Put option): U.S investor, Long (Put option): The other party.)

Net profit/stock $= P - (K - S_T)^+$. (See Discussion 1 Exercise 4, or try to figure this out on your own!)
Net profit/stock $= 3 - (40 - S_T)^+$. (in \$).
From the formula above, we see that it is profitable if

$$3 - (40 - S_T)^+ > 0,$$

or equivalently,

$$(40 - S_T)^+ < 3.$$

- This is always true if $S_T > 40$, since then $40 - S_T < 0$ and $(40 - S_T)^+ = 0$.
- On the other hand, if $S_T < 40$, then $(40 - S_T)^+ = 40 - S_T$. Plug this into the inequality to obtain

  $$40 - S_T < 3,$$

  which simplifies to

  $$S_T > 40 - 3 = 37.$$

From the above analysis, it is profitable for the U.S investor if $S_T > 40$ or ($S_T < 40$ and $S_T > 37$). Combining both of these cases, then we have that it is profitable if

$$\boxed{S_T > 37.}$$

The associated risk would be that if $S_T < 37$, the put option will not be profitable.

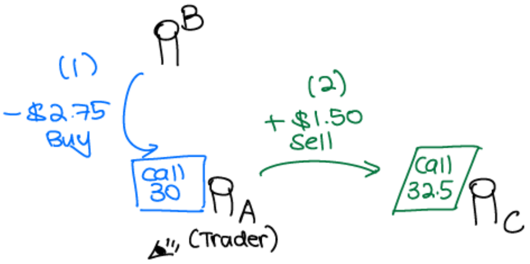
Remark: This can also be obtained using a graphical method, by plotting the graph of the net profit/stock $3 - (40 - S_T)^+$ against S_T, and looking for the range of values of S_T for which the net profit/stock is positive. I'll leave it to you to figure it out (and practice your graph drawing skills).

Exercise 8. (Exercise 2.4.) A stock price is $29. A trader buys one call option contract on the stock with a strike price of $30, and sells a call option contract on the stock with a strike price of $32.50. The market prices of the options are $2.75 and $1.50, respectively. The options have the same maturity date. Describe the trader's position by providing the final payoff function at maturity of the combined position.

Suggested Solution:

Let S_T be the spot price of the stock at maturity. By buying a call option at \$2.75 and selling one at \$1.50, the setup cost is at $\$2.75 - \$1.50 = \$1.25$.

The diagram below shows the setup between a trader and the relevant parties. We label the call option bought with strike price $K_1 = \$30$ as (1), and the call option sold with strike price $K_2 = \$32.50$ as (2). The following table summarizes what happens for three cases: (i) $S_T < 30 = K_1$, (ii) $K_1 = 30 < S_T < 32.5 = K_2$, and (iii) $S_T > 32.5 = K_2$.



Payoff	(1); Trader - Long	(2); Trader - Short	Net Payoff
$S_T < 30$	$S_T < K_1$; **Do not exercise** Payoff = 0	$S_T < K_2$; **Do not exercise** Payoff = 0	Net Payoff = 0
$30 < S_T < 32.5$	$S_T > K_1$; **Exercise** Payoff $= S_T - K_1$ $= S_T - 30$	$S_T < K_2$; **Do not exercise** Payoff = 0	Net Payoff $= S_T - K_1$ $= S_T - 30$
$S_T > 32.5$	$S_T > K_1$; **Exercise** Payoff $= S_T - K_1$ $= S_T - 30$	$S_T > K_2$; **Exercise** Payoff $= K_2 - S_T$ $= 32.5 - S_T$	Net Payoff $= S_T - K_1 + K_2 - S_T$ $= K_2 - K_1$ $= 2.50$

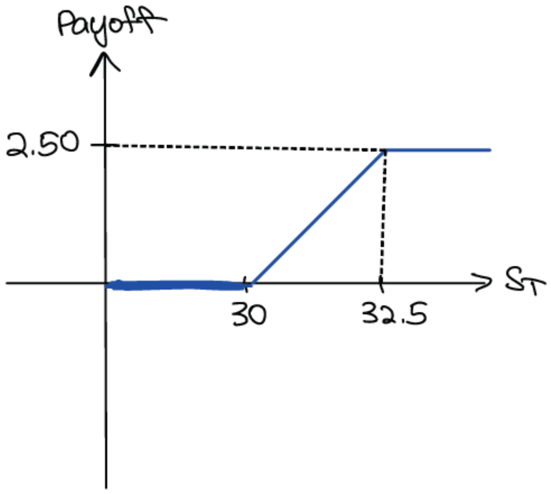
Hence, the payoff function is given by

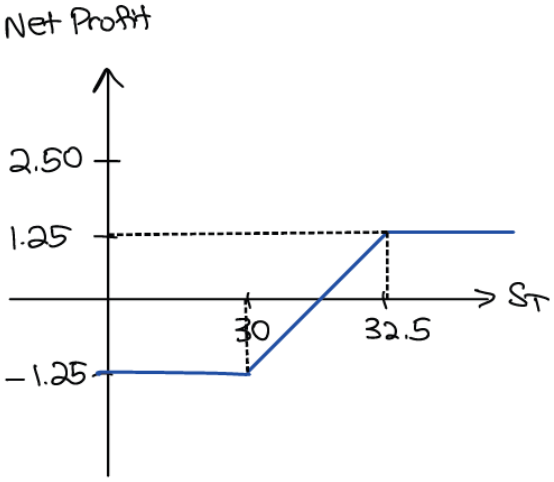
$$\text{Payoff} = \begin{cases} 0 & \text{if } S_T < 30, \\ S_T - 30 & \text{if } 30 < S_T < 32.5, \\ 2.50 & \text{if } S_T > 32.5. \end{cases}$$

The net profit is given by

$$\text{Net Profit} = \text{Payoff} - \text{Setup Cost} = \begin{cases} -1.25 & \text{if } S_T < 30, \\ S_T - 31.25 & \text{if } 30 < S_T < 32.5, \\ 1.25 & \text{if } S_T > 32.5. \end{cases}$$

The following are some graphs of the final payoff and net profits as a function of the spot price of the stock.


Payoff
2.50
30
32.5
S_T


Net Profit
2.50
1.25
−1.25
30
32.5
S_T

Exercise 9. (Exercise 2.5.) Suppose today a corporate treasurer from a U.S. company is saying the following:

"I will have £1,000,000 to sell in 6 months. If the exchange rate (£/\$) is less than 1.19, I want you to give me 1.19. If it is greater than 1.25, I will accept 1.25. If the exchange rate is between 1.19 to 1.25, I will sell the sterling for the exchange rate."

How could you use options written on GBP/USD (£/\$) exchange rate to satisfy the treasurer?

Suggested Solution: Let ER represent the exchange rate (and hence the spot exchange rate at maturity; ie spot price of £ in \$).

- If $\mathrm{ER} < 1.19$, then the treasurer wants the other party to "eat the loss" incurred. This implies that the treasurer must be **given the option** to sell £. Hence, I would **buy a put option** with **strike exchange rate** 1.19.
- If $\mathrm{ER} > 1.25$, the treasurer is okay with the other party benefiting from it (since the treasurer is willing to only sell it for 1.25). Hence, as an "efficient market", I would thus **sell a call option** the other party with **strike exchange rate** 1.25 so that it would be exercised by the other party once $\mathrm{ER} > 1.25$.

3 Discussion 3

Future Contracts and Arbitrage.

Definitions/Terminologies:

- An **arbitrage** involves taking offsetting positions in two or more financial instruments to lock in a **riskless profit**.
- An **arbitrage** opportunity is a trading strategy that has **no upfront costs/value** and **always leads to a non-negative (riskless) profit**.
- A **futures contract** is an agreement between two parties to buy or sell an asset for a predetermined delivery price (**futures price**) at a predetermined future time (**maturity date**).
- A **margin** is a cash balance deposited by an investor with their broker in a **margin account**.
  (Margin approximately means small $\sim$ usually a small fraction of the futures price.)
- An **initial margin** is the amount provided by the broker to enter (initialize) a futures contract
- A **margin call** happens if the margin account falls below the **maintainance margin** and the broker deposits **variation margin** to top off the margin account to the level of the initial margin.
  (Since the exchange side "calls" the broker to ask them to top up their margin account.)

Typically, a futures contract is traded on an **exchange**, with brokers required to register for a margin account with the exchange's **clearing house** (that facilitates settlement).

A key difference between the forward and futures contracts is the **daily settlement** of the futures contract. The brief mechanism of this with an example as follows. Suppose that the current futures price is $1,000, with $100 in the margin account (as the initial margin) and a maintenance margin of $60. Furthermore, the trader is in a long position in the futures contract.

For each new day, we have:

	Futures Price	Margin Account
Day 1	$1000	$100
Update: Futures Price +$50		
Day 2	+$50 =$1,050	+$50 =$150
Update: Futures Price -$80		
Day 3	−$80 =$970	−$80 =$70
Update: Futures Price -$20		
Day 4	−$20 =$950	−$20 =$50 < $60
Margin Call: Top up $50		
Day 4	=$950	+$50 =$100
Update: Futures Price +$20		
Day 5	+$20 =$970	+$20 =$120
Close Out		
Update: Futures Price +$120		
Day 6	+$20 =$990	=$120

In the above example, the futures contract is **closed out** at the end of Day 5. Daily settlement updates the margin account in accordance with the change in the future price as compared to the **previous day**. Closing out here refers to "closing the account to future updates in futures prices".

In practice, "closing out" here is done by selling the futures contract (if you're in a long position), or buying the required deliverable at the current futures price (if you're in a short position so that you have technically "paid" for it upfront). The payoff here can be computed using

$$\text{Payoff} = \text{Futures Price in Day 5} - \text{Future Prices in Day 1} = -\$30.$$

Note that this is equivalent to computing the net change in your margin account, accounting for the additional amount of money topped up through margin calls.

$$\text{Payoff} = \text{Margin Account on Day 5} - \text{Margin Account on Day 1} - \text{Top up} = \$120 - \$100 - \$50 = -\$30.$$

In fact, the following formulas compute the profit/loss in the relevant positions for a futures contract. If the futures price when the contract was entered into at t_1 is F_{t_2} and the futures price when the contract is closed out at t_2 is F_{t_2}, we have

- Profit/Loss for **long** position $= F_{t_2} - F_{t_1}$.
- Profit/Loss for **short** position $= F_{t_1} - F_{t_2}$.

More terminologies are included in the comparison of forward and futures contracts, supplemented in relevant footnotes as shown below:

Forward	**Futures**
Private contract	Traded on an exchange
Not standardized	Standardized contract
Usually one specified delivery date	Range of delivery dates[8]
Settled at the end of the contract	Daily settlement
Delivery or final cash settlement	Usually closed out prior to maturity
Some credit risk	Virtually no credit risk[9]

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[8] In fact, the term **delivery month** is the last month in which the seller (short) must deliver, and the buyer (long) must accept/pay. For this to happen, the seller (long position) must issue a "**notice of intent to deliver**" to the exchange clearing house.

[9] This is because any losses are settled immediately on a daily basis through an "initial deposit" in the margin account.

Exercise 10. (Exercise 3.1.) The price of gold is currently $1,200 per ounce. The forward price for delivery in 1 year is $1,300 per ounce. An arbitrageur can borrow money at 3% per annum (= per year). What should the arbitrageur do? Assume that the cost of storing gold is zero and that gold provides no income.

Suggested Solutions:

Observe that if one could just buy the gold at $1,200 per ounce and immediately enter a forward contract to sell (short) the gold at the forward price of $1,300 per ounce, one would earn $100 per ounce. This is **almost an arbitrage opportunity**, except that one has to account for the fact that the amount of money used to purchase gold can be used to earn interest/one has to pay interest to obtain the correct amount from the bank. It might be possible that the $100/ounce is not worth it if this is greater than the cost of borrowing.

Hence, to show that this is **indeed an arbitrage opportunity**, observe that the final profit per ounce of gold is computed as follows:

$$1,300 - \underbrace{1,200e^{0.03}}_{\text{Interest Rates on Loans}} = +63.45.$$

Hence, the arbitrageur will borrow money to buy as much gold as possible and enter a forward contract in the short position to sell all of them at the forward price.

Observe that no risks are involved; the arbitrageur simply has to wait for the contract to mature, collect profits at maturity, and pay the bank the amount that they borrowed. Furthermore, there are no upfront costs as the arbitrageur borrows the money required (and even by borrowing, this still leads to a profit).

Exercise 11. (Exercise 3.2.) Suppose that you enter into a short futures contract to sell July silver for \$17.20 per ounce. The size of the contract is 5,000 ounces. The initial margin is \$4,000, and the maintenance margin is \$3,000. What change in the futures price will lead to a margin call? What happens if you do not meet the margin call?

Suggested Solutions:

- If the balance in the margin account falls by \$4,000 - \$3,000 = \$1,000, then a margin call will be required. This is equivalent to making a loss of more than \$1,000. This is equivalent to making a loss of more than \$0.20/ounce of silver.

  In a short position for a futures contract, you will be required to deliver silver for \$17.20/ounce. Hence, a loss is incurred if the futures price of silver increases (since you will be making a loss by selling it at a lower price). Hence, from the previous bullet point, a margin call happens if (F denotes the futures price leading to a margin call)

  $$F_{t_1} - F_{t_2} = 17.20 - F < -0.20$$

  or equivalently,

  $$F > 17.40.$$

- If a margin call is not met, **the broker will close out the position**. (Hull Chapter 2.4.)

Exercise 12. (Exercise 3.3.) Suppose that in September 2021 a company takes a long position in a contract on May 2022 crude oil futures. It closes out its position in March 2022. The futures price (per barrel) is $48.30 when it enters into the contract and $50.50 when it closes out its position. One contract is for the delivery of 1,000 barrels. What is the company's total profit? When is it realized?

Suggested Solutions:

- Let F_{t_2} be the futures price per barrel on March 2022 and F_{t_1} be the futures price per barrel when a company first enters into a futures contract.

  For 1,000 barrels, profit (Long Position) = \$$(F_{t_2} - F_{t_1}) \times 1000 =$ \$(50.50 − 48.30) × 1000 = \$2,200.

- The profit is **immediately realized** when the contract closes out.

Exercise 13. (Exercise 3.4.) It is July 2021. A mining company has just discovered a small deposit of gold. It will take 6 months to construct the mine. The gold will then be extracted on a more or less continuous basis for 1 year. Futures contracts on gold are available with delivery months every 2 months from August 2021 to December 2022. Each contract is for the delivery of 100 ounces. Discuss how the mining company might use futures markets for hedging.

Suggested Solutions:

- Since the mine was only discovered in July 2021 and it takes 6 months to construct the mine, gold will only be produced starting January 2022.
- The mining company can estimate its production on a month-to-month basis. It can then short futures contracts to lock in the price received for the gold to hedge against price fluctuations in gold prices.
- Since the delivery months are available every 2 months, for every 2 months starting from February 2022, if the mining company expects a total of about $100X$ ounces of gold to be extracted within 2 months, then the mining company should **short** X gold futures contracts for the upcoming delivery month.

Exercise 14. (Exercise 3.5.) What position is equivalent to a long forward contract to buy an asset at K on a certain date T and a put option to sell it for K on that date?

Hint: Determine the payoff of the combined position (long forward plus long put) and identify how this payoff can be created by a different position.

Suggested Solutions: The payoff of the combined position is given by

$$\begin{aligned}
\text{Payoff} &= \underbrace{(S_T - K)}_{\text{Long Forward}} + \underbrace{(K - S_T)^+}_{\text{Long Put}} \\
&= \begin{cases} 0 & \text{if } K > S_T, \\ S_T - K & \text{if } K < S_t, \end{cases} \\
&= (S_T - K)^+ \\
&= \text{ Payoff of } \textbf{Long Call}.
\end{aligned}$$

Exercise 15. (Exercise 3.6.) Consider a trader who is trading call and put options with maturity $T > 0$ on a stock.

(i) Provide and draw the payoff profile at maturity T as a function of the stock price S_T at time T of following trading strategies:

(a) *Butterfly Spread:* This position consists of call options with three different strikes but with the same maturity $T > 0$. Specifically, the trader buys two call options with strikes K_1 and K_2 where $K_2 > K_1$, and sells/writes two calls with strike $\frac{K_1+K_2}{2}$.

(b) *Straddle:* The trader has a long position in a call option and a put option for the same stock with the same strike and the same maturity.

(ii) Draw the trader's net profit at maturity as a function of the price of the underlying stock S_T at time T of both trading strategies (a) and (b). In particular, for implementing the *butterfly spread strategy* does the trader have to pay money, or is she receiving money? Explain carefully.

(i) (a)

- (1); Long Call with strike price K_1.
- (3); Long Call with strike price K_2.
- (2); Long Put with strike price $\frac{K_1+K_2}{2}$.

Payoff	(1); Long Call	2×(2); Short Call	(3); Long Call	Net Payoff
$S_T < K_1$	$S_T < K_1$ **Do not Exercise;** Payoff $= 0$	$S_T < \frac{K_1+K_2}{2}$ **Do not Exercise;** Payoff $= 0$	$S_T < K_2$ **Do not Exercise;** Payoff $= 0$	0
$K_1 < S_T < \frac{K_1+K_2}{2}$	$S_T > K_1$ **Exercise;** Payoff $= S_T - K_1$	$S_T < \frac{K_1+K_2}{2}$ **Do not Exercise;** Payoff $= 0$	$S_T < K_2$ **Do not Exercise;** Payoff $= 0$	$S_T - K_1$
$\frac{K_1+K_2}{2} < S_T < K_2$	$S_T > K_1$ **Exercise;** Payoff $= S_T - K_1$	$S_T > \frac{K_1+K_2}{2}$ **Exercise;** Payoff $= 2(\frac{K_1+K_2}{2} - S_T)$	$S_T < K_2$ **Do not Exercise;** Payoff $= 0$	$K_2 - S_T$
$S_T > K_2$	$S_T > K_1$ **Exercise;** Payoff $= S_T - K_1$	$S_T > \frac{K_1+K_2}{2}$ **Exercise;** Payoff $= 2(\frac{K_1+K_2}{2} - S_T)$	$S_T > K_2$ **Exercise;** Payoff $= S_T - K_2$	0

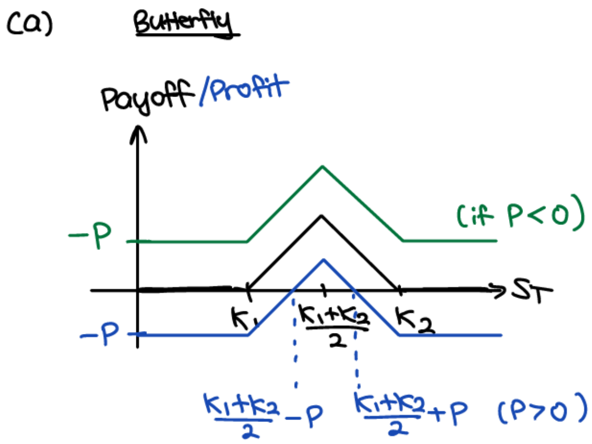
(b)

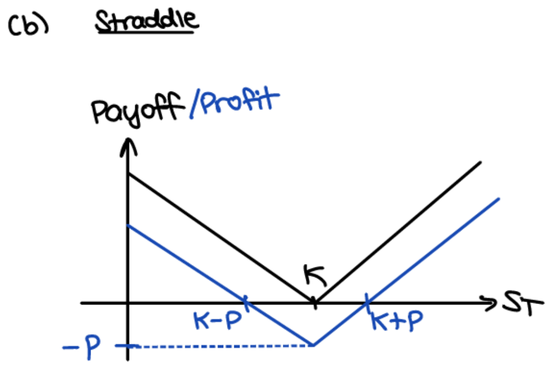
Payoff	Long Call	Long Put	Net Payoff
$S_T < K$	$S_T < K$ **Do not Exercise;** Payoff $= 0$	$K > S_T$ **Exercise;** Payoff $= K - S_T$	$K - S_T$
$S_T > K$	$S_T > K$ **Exercise;** Payoff $= S_T - K$	$K < S_T$ **Do not Exercise;** Payoff $= 0$	$S_T - K$

$$\text{Payoff} = \begin{cases} K - S_T & \text{if } K > S_T \\ S_T - K & \text{if } S_T > K \end{cases} = |K - S_T| = |S_T - K|.$$

Here, $|\cdot|$ refers to the absolute value function.

(ii) Let P be the premium paid on all of the options in each strategy above. In the straddle strategy, since the trader purchases both options, the premium paid will be positive. However, for the butterfly strategy, since selling call options net revenue but buying call options net losses, one would have to be careful. In this case, if the startup cost is positive, we assume that $P > 0$ (and $P < 0$ for a negative startup cost). The relevant profit functions are drawn below.


(a) Butterfly
Payoff / Profit
−P
(if P<0)
−P
S_T
K_1
$\frac{K_1+K_2}{2}$
K_2
$\frac{K_1+K_2}{2}-P$
$\frac{K_1+K_2}{2}+P$
(P>0)


(b) Straddle
Payoff / Profit
K
S_T
K−P
K+P
−P

4 Discussion 4

Hedging Strategies using Futures
Definitions/Terminologies:

- **Hedging** is an investment that is implemented solely to reduce or cancel out the financial risk in another investment.
  (To *hedge against* is to “protect against losses/risk”.)
- **Cross hedging** (using futures) refers to hedging an asset that may not be the same as the asset underlying the futures contract.
- The **basis** in a hedging situation is defined as

$$\text{Basis} = \text{Spot price of asset to be hedged} - \text{Futures price of contract used.}$$

- The **basis risk** is the risk to a hedger arising from uncertainty about the **basis when the hedge is closed out**.
- The **hedge ratio** is defined as

$$\text{Hedge Ratio} = \frac{\text{Size of the position in a hedging instrument (Futures contract)}}{\text{Size of the position being hedged (Exposure)}}.$$

  (For a hedge ratio of 2, to “protect against losses” for 1 unit of apple, I “buy/sell” 2 units of oranges.)
  (Exposure $\sim$ (original) “assets exposed to changes”)
- The **minimum variance hedge ratio** (h^*) is the hedge ratio that minimizes variance of the value of the hedged position.

$$\text{Minimum Variance Hedge Ratio} := h^* = \rho \times \frac{\sigma_S}{\sigma_F}. \tag{1}$$

  Here,
  - ΔS := Change in spot price ($\sim$ to be “hedged”),
  - ΔF := Change in futures price ($\sim$ used to “hedge”),
  - σ_S refers to the standard deviation of ΔS,
  - σ_F refers to the standard deviation of ΔF, and
  - ρ refers to the correlation coefficient between ΔS and ΔF.
- A **stock index** tracks changes in the value of a hypothetical portfolio of stocks.
  (Futures contract on stocks is used for speculation and hedging against equity (stock) prices.)

More on basis (mainly for cross-hedging):

- The basis risk consists of closing out the futures contract long before its delivery month (say t_2), and cross hedging.
- Basis $\approx$ "cost associated with hedging using another asset (cross-hedging)"
- **Short** Hedge:
  - Want: **Sell** an asset (at time t_2 with spot price S_{t_2}).
  - Hedge by: **Short** position in futures ("**Selling** a related asset").
  - Basis = $\underbrace{S_{t_2}}_{\text{Selling the asset}} - \underbrace{F_{t_2}}_{\text{Buying the asset underlying futures}}$.
    (This is so that we are "effectively trading with (selling)" the underlying asset on the futures instead of the asset itself.)
  - Effective price obtained to **sell** the asset:

$$\begin{aligned}
\text{Effective price of selling} &= \text{Spot selling price} + \text{“Profit” from Futures} \\
&= S_{t_2} + F_{t_1} - F_{t_2} \\
&= F_{t_1} + (S_{t_2} - F_{t_2}) \\
&= F_{t_1} + b_{t_2} \\
&= \text{Price locked in at } t_1 \text{ via Futures} + \text{Basis upon closing out.}
\end{aligned}$$

- **Long** Hedge:
  - Want: **Buy** an asset (at time t_2 with spot price S_{t_2}).
  - Hedge by: **Long** position in futures ("**Buying** a related asset").
  - Basis = $\underbrace{S_{t_2}}_{\text{Buying the asset}} - \underbrace{F_{t_2}}_{\text{Selling the asset underlying futures}}$.
    (This is so that we are "effectively trading with (buying)" the underlying asset on the futures instead of the asset itself.)
  - Effective price to pay to **get** the asset:

$$\begin{aligned}
\text{Effective price of buying} &= \text{Spot buying price} + \text{“Losses” from Futures} \\
&= S_{t_2} + -(F_{t_2} - F_{t_1}) \\
&= F_{t_1} + (S_{t_2} - F_{t_2}) \\
&= F_{t_1} + b_{t_2} \\
&= \text{Price locked in at } t_1 \text{ via Futures} + \text{Basis upon closing out.}
\end{aligned}$$

Exercise 16. (Exercise 4.1.) It is now June. A company knows that it will sell 5,000 barrels of crude oil in September. It uses the October CME Group futures contract to hedge the price it will receive. Each contract is on 1,000 barrels of "light sweet crude". What position should it take? What price risks is it still exposed to after taking the position?

Suggested Solutions:

- **Short Hedge**: Short 5 × October CME Group futures on "light sweet crude".
- Price risk: By agreeing to be in the short position for the futures contract, the company "locks in" the October futures price. Since

  $$\text{Effective price of buying} = \text{October futures price} + \text{Basis upon closing out (September)},$$

  the company is still susceptible to **basis risk** (accounting for the fixed October futures price).

  Hence, the effective buying price can be higher (than the expected October futures price) if the basis upon closing out is positive.

Exercise 17. (Exercise 4.2.) Sixty futures contracts are used to hedge an exposure to the price of silver. Each futures contract is on 5,000 ounces of silver. At the time the hedge is closed out, the basis is \$0.20 per ounce. What is the effect of the basis on the hedger's financial position if

(a) The trader is hedging the purchase of silver, and

(b) The trader is hedging the sale of silver.

Suggested Solutions:

(a) If the trader is hedging the purchase of silver, it is in a **long hedge**. This implies that the trader will be in the long position of the futures contracts. Since the

$$\text{Effective price of buying} = \text{Futures price} + \text{Basis upon closing out},$$

the effective price of buying is increased (relative to the futures price) by \$0.20 per ounce, or a total of

$$\$0.20 \text{ per ounce } \times 5000 \text{ ounces/contract } \times 60 \text{ contracts } = \$60{,}000.$$

This is incurred as a **loss** to the hedger. Hence, the net profit/loss is $\boxed{-\$60{,}000}$.

Remark: This is relative to what the hedger "expects" to only pay for, which is the futures price adjusted for the number of ounces of silver.

(b) If the trader is hedging the sale of silver, it is in a **short hedge**. This implies that the trader will be in the short position of the futures contracts. Since the

$$\text{Effective price of selling} = \text{Futures price} + \text{Basis upon closing out},$$

the effective price of selling is increased (relative to the futures price) by \$0.20 per ounce, or a total of

$$\$0.20 \text{ per ounce } \times 5000 \text{ ounces/contract } \times 60 \text{ contracts } = \$60{,}000.$$

This is incurred as a **gain** to the hedger. Hence, the net profit/loss is $\boxed{+\$60{,}000}$.

Exercise 18. (Exercise 4.3.) Suppose you are a website designer who prefers to be paid in cryptocurrency Bitcoin rather than in USD.

On March 15, 2022, you make a deal with a client to create their website with a deadline at the end of July 2022. Your client agrees to pay you 0.30 Bitcoin for your work, but the client insists on paying you 0.30 Bitcoin end of July at the deadline, and not earlier. Knowing that Bitcoin is very volatile, the arrangement of being paid 0.30 Bitcoin only in July but not on March 15 (in advance) makes you feel a bit worried. However, Micro Bitcoin Futures (MBT) are traded on the exchange CME. The contract size is 0.10 units of Bitcoin per contract, the initial margin is 50% of the size (in USD) of the total position taken in the futures contracts, and the maintenance margin is $1,255 per contract. The futures contracts are settled in cash (in USD) on the last day of trading of the delivery month and do not involve any "physical" (digital) transfer of Bitcoin.

On March 15, the spot price of Bitcoin quoted on the cryptocurrency exchange Binance was $38,400 per one unit of Bitcoin and in July 2022 Micro Bitcoin Futures are traded for $38,800 per one unit of Bitcoin on CME. On this day, you decide to use July 2022 Micro Bitcoin Futures for hedging your entire exposure to price changes of Bitcoin.

(a) Describe the position you have to take with the July Bitcoin futures contracts from CME in order to hedge yourself. What is the initial margin you have to provide?

Friday, July 29, 2022, is the last day of trading July Bitcoin Futures on CME, and at the end of that day your position in the futures contracts is settled in cash in USD at $23,000 per Bitcoin, the final settlement price determined by CME (the CME Bitcoin Reference Rate on this day). Moreover, you also finish your work on this day and you receive 0.30 Bitcoin from your client.

(b) On Friday, July 29, 2022, the Bitcoin spot price quoted on Binance is $23,900 per Bitcoin. If you decide to sell the 0.3 Bitcoin you receive from your client on Binance for this spot price, what is the total USD amount you receive for your Bitcoins, together with taking into account your hedge from the futures contracts?

Suggested Solutions:

(a) Short hedge; short position on the futures contracts.

Initial margin = 50% of $38,800 /Bitcoin × 0.30 bitcoin = $\boxed{\$5{,}820}$. Here, we use $38,800 /Bitcoin since this is the quoted futures price on March 15.

(b) The amount received by the trader *per Bitcoin* is the sum of the spot price at maturity and the profit from the futures contract. Observe that

- Spot price at maturity = $23,900.
- Profit from futures = $38,800 − $23,000 = $15,800.
- Hence, the total amount obtained = $23,900 + $15,800 = $39,700.

Remark: One can also compute this by recalling that

$$\text{Effective price of selling} = \text{Futures price} + \text{Basis upon closing out.}$$

The futures price is at $38,800 per bitcoin, and the basis upon closing out is given by $S_{t_2} - F_{t_2} = \$23{,}900 - \$23{,}000 = \$900$. Hence, you would receive $38,800 + $900 = $39,700.

Effectively, the exposure here is the Bitcoin to be sold on Binance, and hedging is done by entering a futures contract on the exchange CME.

Note that the total amount received would be 0.3 × $39,700/Bitcoin = $\boxed{\$11{,}910}$.

Exercise 19. (Exercise 4.4.) A trader owns 55,000 units of a particular asset and decides to hedge the value of her position with futures contracts on another related asset. Each futures contract is on 5,000 units. The spot price of the asset that is owned is \$28 and the standard deviation of the change in this price over the life of the hedge is estimated to be \$0.43. The futures price of the related asset is \$27 and the standard deviation of the change in this over the life of the hedge is \$0.40. The coefficient of correlation between the spot price change and the futures price change is estimated to be 0.95.

(a) What is the minimum variance hedge ratio?

(b) Should the hedger take a long or short futures position?

(c) What is the optimal number of futures contracts (when adjustments for daily settlement are not considered)?

Suggested Solutions:

(a) From the question, we have

- $\sigma_S = 0.43$,
- $\sigma_F = 0.40$,
- $\rho = 0.95$.

Hence, by (1), we have

$$h^* = \rho \times \frac{\sigma_S}{\sigma_F} = 0.95 \times \frac{0.43}{0.40} = \boxed{1.02125}.$$

(b) Since the trader owns 55,000 units of a particular asset, they should take a short futures position (short hedge, $\sim$ “selling” the assets).

(c)

$$\begin{aligned}&\text{Optimal number of units of the hedging (futures) asset}\\ &= \text{Minimum Variance Hedge Ratio} \times \text{Number of units of hedged asset}\\ &= 1.02125 \times 55,000 = 56168.75.\end{aligned}$$

Hence, the optimal number of futures contracts $= \frac{56168.75 \text{ units}}{5000 \text{ units/contract}} = 11.23375 \approx \boxed{11}$.

Exercise 20. (Exercise 4.5.) A company wishes to hedge its exposure to a new fuel whose price changes have a 0.6 correlation with gasoline futures price changes. The company will lose \$1,000,000 for each 1 cent increase in the price per gallon of the new fuel over the next three months. The new fuel's price changes have a standard deviation that is 50 % greater than price changes in gasoline futures prices.

(a) If gasoline futures are used to hedge the exposure, what should the hedge ratio be?

(b) What is the company's exposure measured in gallons of the new fuel? What position, measured in gallons, should the company take in gasoline futures?

(c) How many gasoline futures contracts should be traded? Each contract is on 42,000 gallons.

Suggested Solutions:

(a) This would be the minimum variance hedge ratio h^*. From the question, we have

- Exposure/Hedged asset: New Fuel.
  Hedging asset on futures: Gasoline.
- $\sigma_S = 1.5\sigma_F$, and
- $\rho = 0.6$.

By (1), we have

$$h^* = \rho \times \frac{\sigma_S}{\sigma_F} = 0.6 \times \frac{1.5\sigma_F}{\sigma_F} = \boxed{0.9}.$$

(b) Since the company will lose \$1,000,000 for each 1 cent increase in the price per gallon, then the number of gallons the company has (ie the company's exposure) is given by

$$\frac{\text{\$1,000,000}}{\text{\$0.01/ gallon}} = \text{100,000,000 gallon } (100 \text{ million gallons}).$$

Since this is recorded as a loss, this implies that the company is trying to buy the new fuel.

Hence, the company's position should be to long $h^* \times$100,000,000 $=$ $\boxed{\text{90,000,000}}$ gallons of gasoline.

(c) Number of contracts $= \frac{\text{90,000,000 gallons}}{\text{42,000 gallons/contract}} =$ 2142.857 $\approx$ $\boxed{2143}$.

Exercise 21. (Exercise 4.6.) The following table provides historical data on $n = 10$ monthly changes in the spot price ΔS and the future price ΔF for a certain commodity. Use the data to calculate a minimum variance hedge ratio.

ΔS	+0.50	+0.61	−0.22	−0.35	+0.79	+0.04	+0.15	+0.70	−0.51	−0.41
ΔF	+0.56	+0.63	−0.12	−0.44	+0.60	−0.06	+0.01	+0.80	−0.56	−0.46

Hint: Recall that for a given sample $x_1, \cdots, x_n$, the sample mean $\bar{x}$ and the sample standard deviation s_x can be computed via

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad s_x = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}.$$

Moreover, given a series of n measurements of pairs (x_i, y_i), $i = 1, \cdots, n$, the sample correlation coefficient is computed via

$$\rho_{x,y} = \frac{\sum_{i=1}^{n}(x_i - \bar{x}) \cdot (y_i - \bar{y})}{(n-1) \cdot s_x \cdot s_y}.$$

Suggested Solutions:

This is an exercise of computing estimates for each of the statistical quantities in the formula for minimum variance hedge ratio. Using a calculator, we have

ΔS	+0.50	+0.61	−0.22	−0.35	+0.79	+0.04	+0.15	+0.70	−0.51	−0.41
ΔF	+0.56	+0.63	−0.12	−0.44	+0.60	−0.06	+0.01	+0.80	−0.56	−0.46
$\overline{\Delta S} = +0.130$ $\overline{\Delta F} = +0.096$										
$\Delta S - \overline{\Delta S}$	+0.37	+0.48	−0.35	−0.48	+0.66	−0.09	+0.02	+0.57	−0.64	−0.54
$\Delta F - \overline{\Delta F}$	+0.464	+0.534	−0.216	−0.536	+0.504	−0.156	−0.086	+0.704	−0.656	−0.556
$\sum_{i=1}^{10}(\Delta S_i - \overline{\Delta S})^2 = 2.1904$, $s_S = 0.49333$, $\sum_{i=1}^{10}(\Delta F_i - \overline{\Delta F})^2 = 2.3352$, $s_F = 0.51156$.										
$(\Delta S - \overline{\Delta S})(\Delta F - \overline{\Delta F})$	+0.1717	+0.2563	+0.0756	+0.2573	+0.3326	+0.0140	−0.0017	+0.4013	+0.4198	+0.3002
$\sum_{i=1}^{10}(\Delta S_i - \overline{\Delta S}) \cdot (\Delta F_i - \overline{\Delta F}) \approx 2.2271$, $\rho \approx 0.98$.										

Since s_S and s_F are (unbiased) estimators of σ_S and σ_F, we have by (1),

$$h^* \approx \frac{\rho \times s_S}{s_F} \approx \boxed{0.945}.$$

5 Discussion 5

Interest Rates and Related Concepts
Definitions/Terminologies:

- An **interest rate** in a particular situation defines the amount of money a borrower promises to pay the lender. This is equivalent to the concept of **the time value of money**.
- Compounding and Discounting.
  - Interest rate (per year/per annum/p.a): r,
  - Investment period (in years): n, and
  - Compounding frequency (per year): m (for simple compounding/discounting).

  (i) Simple:
  $$\text{Value}(n) = \text{Value}(0) \times \left(1 + \frac{r}{m}\right)^{m \cdot n}$$
  (ii) Continuous:
  $$\text{Value}(n) = \text{Value}(0) \times e^{r \cdot n}$$

  Simple/Continuous compounding/discounting is the terminal (future)/current value of an amount A invested/compounded for n years at a rate of r p.a. compounded m times per year/continuously.
- The n**-year zero-coupon interest rate** is the **rate of interest earned** on an investment that starts today ($t = 0$) and lasts for n years (till $t = n$), with **no intermediate payments (zero coupons)** (that is, all the interest and principal is realized at the end of n years).

  Also known as:
  - n**-year spot rate** ($\sim$ today's rate for $n-$year investment), and
  - n**-year zero rate** ($\sim$ today's rate with zero coupons (interest)).
- A **bond** is an instrument of indebtedness of the bond issuer to the holders in which the issuer owes (and hence is "bonded to") the holders a debt and is obliged to pay them interest (the **coupons**) and the **principal** (face value) at a later date (**maturity date**).
  - Zero-coupon bonds = No coupons (no interest).
  - Theoretical price/Bond price = **Present (discounted) value** of all future cash flows.
  - Bond yield = **Single discount rate** such that
    $$\text{Present (discounted) value of all cash flows} = \text{Market Price}.$$
  - A **zero(-coupon yield) curve** is a chart showing the annual zero rates (spot rates) as a function of maturity.
  - The **duration of the bond** D is defined mathematically by
    $$D = \frac{\sum_{i=1}^{n} t_i \cdot c_i \cdot e^{-y \cdot t_i}}{B}, \tag{2}$$
    where
    - B refers to the bond price,
    - c_i refers to a future cash flow coming in at time t_i
    - with a discount factor $e^{-y \cdot t_i}$ using the bond yield y.
    - We also denote the sum $\sum_{i=1}^{n} t_i \cdot c_i \cdot e^{-y \cdot t_i}$ as the weighted sum of future cash flows.
  - **Duration Relationship**: The change in bond price ΔB and the change of bond yield Δy are **approximately related** via the following equation:
    $$\Delta B \approx -B \cdot D \cdot \Delta y. \tag{3}$$
    Note: ΔB and Δy are of different sign; since a higher yield would cause the bond price to decrease (since the present value of all future cash flows are discounted further).

- **Forward interest rates** are the rates of interest implied by current zero rates for periods of time in the future.

  (The definition is a little confusing without understanding how it works. Hence, we will look at the following computational example.)

  Consider today's spot rates:

Year (n)	Today's spot rate $r_0(n)$ p.a.	Forward Rate $f_0(n-1,n)$ p.a.
1	3.0	(N.A.)
2	4.0	5.0
3	4.6	5.8

  Intuitively, "to get a 2−year interest rate up to 4% from the 1−year interest rate of 3%, we must have (this implies) that the interest rate on the second year (ie the **forward interest rate** from year 1 to year 2) must be at 5%".

  Mathematically (with T_1 and T_2 measured in years, with $T_2 > T_1$),

$$\underbrace{f_0(T_1,T_2)}_{\text{RHS implies this I.R from } T_1 \text{ to } T_2} = \frac{\underbrace{r_0(T_2)\cdot T_2}_{\text{I.R for } T_2 \text{ years starting today}} - \underbrace{r_0(T_1)\cdot T_1}_{\text{I.R for } T_1 \text{ years starting today}}}{T_2 - T_1}. \tag{4}$$

  Example: $f_0(3,2) = \frac{4.6\times 3 - 4.0\times 2}{3-2} = 5.8$.

- A **forward rate agreement (FRA)** is an over-the-counter contract (agreement) to exchange a predetermined fixed rate R_K for a reference rate R_M (floating rate) that will be observed in the market at a future time.

  Both rates are applied to a specified principal amount L for a specified future time period from T_1 to T_2. (This is done via simple compounding.)

  Floating rates: LIBOR (London Interbank Offered Rate) or SOFR (Secured Overnight Financing Rate).

  Strategy: Draw an analogy to forward contracts.
  Borrow from the rate that you are in the **long** position for.

  Scenario:

  - Long position in FRA:
    - Long: Reference Rate R_K.
    - To do so: Borrow L from R_K, lend L to R_M.
    - Remark: The rate that you are in the long position for $\sim$ Borrow from that!
    - Payoff at Maturity:

$$\text{Payoff} = \underbrace{-L\cdot R_K\cdot(T_2-T_1)}_{\text{To return at } R_K} + \underbrace{L\cdot R_M\cdot(T_2-T_1)}_{\text{To profit at } R_M} = L\cdot(R_M-R_K)\cdot(T_2-T_1)$$

    - Hedge against **increasing** interest rates.
      (Payoff > 0 if $R_M > R_K$ i.e. floating rate is higher than the fixed rate.)
  - Short position in FRA:
    - Short: Reference Rate R_K.
    - To do so: Borrow L from R_M, lend L to R_K.
    - Payoff at Maturity:

$$\text{Payoff} = \underbrace{-L\cdot R_M\cdot(T_2-T_1)}_{\text{To return at } R_M} + \underbrace{L\cdot R_K\cdot(T_2-T_1)}_{\text{To profit at } R_K} = L\cdot(R_K-R_M)\cdot(T_2-T_1)$$

    - Hedge against **decreasing** interest rates.
      (Payoff > 0 if $R_M < R_K$ i.e. floating rate is lower than the fixed rate.)

Exercise 22. (Exercise 5.1.) An interest rate is quoted as 5% per annum with semiannual compounding. What is the equivalent rate (p.a.) with

(a) Annual compounding,

(b) Monthly compounding, and

(c) Continuous compounding?

Suggested Solutions:
Semiannual compounding compounds every half a year with a compounding factor of $1 + 5\%/2 = 1 + \frac{5/2}{100} = 1.025$. In one year, the overall compounding factor is $1.025^2 = 1.050625$.

(a) Annual compounding compounds once a year. Suppose that the p.a rate is r. The corresponding compounding factor is $(1 + r)$. For this to be equal to the aforementioned semiannual compounding rate, we require

$$1 + r = 1.050625,$$

or equivalently,

$$r = 0.050625 \approx 5.06\%.$$

(b) Monthly compounding compounds once a month. Suppose that the p.a rate is r. The corresponding compounding factor is $(1+r/12)$. This is compounded 12 times in a year. Hence, the overall compounding factor is $(1+r/12)^12$. For this to be equal to the aforementioned semiannual compounding rate, we require

$$(1 + r/12)^{12} = 1.050625.$$

This can be solved to give

$$r = 12((1.050625)^{1/12} - 1) = 0.049487 \approx 4.95\%.$$

(c) For continuous compounding, the overall compounding factor is given by $e^{r \cdot n}$ for n years, or just e^r for one year (with a p.a rate of r). For this to be equal to the aforementioned semiannual compounding rate, we require

$$e^r = 1.050625,$$

or

$$r = \ln(1.050625) = 0.0493852 \approx 4.94\%.$$

Exercise 23. (Exercise 5.2.) The table below gives Treasury zero rates and cash flows on a Treasury bond. Zero rates are continuously compounded.

(a) What is the bond's theoretical price?

(b) What is the bond's yield assuming it sells for its theoretical price?

Maturity (years)	Zero rate (p.a.)	Coupon payment	Principal
0.5	2.0%	\$20	
1.0	2.3%	\$20	
1.5	2.7%	\$20	
2.0	3.2%	\$20	\$1,000

Suggested Solutions:

(a) Recall that this is the present value of all future cash flow. The discounted value of each cash flow (in \$) is given as follows:

Years	Cash Flow Source	Cash Flow	Discounted Value
0.5	Coupon	20	$+20e^{-0.020\times0.5}$
1.0	Coupon	20	$+20e^{-0.023\times1}$
1.5	Coupon	20	$+20e^{-0.027\times1.5}$
2.0	Coupon and Principal	1020	$+1020e^{-0.032\times2}$
Bond Price (Sum the last column)			$+1015.32.$

(b) Recall that this is the **single discount rate** to be used in our computation. Hence, we have

Years	Cash Flow Source	Cash Flow	Discounted Value
0.5	Coupon	20	$+20e^{-y\times0.5}$
1.0	Coupon	20	$+20e^{-y\times1}$
1.5	Coupon	20	$+20e^{-y\times1.5}$
2.0	Coupon and Principal	1020	$+1020e^{-y\times2}$
Bond Price (Sum the last column)			$20(e^{-0.5y}+e^{-y}+e^{-1.5y})+1020e^{-2y}.$

Solve

$$20(e^{-0.5y}+e^{-y}+e^{-1.5y})+1020e^{-2y}=1015.32$$

numerically to obtain $y\approx0.031777\approx3.18\%$.

Exercise 24. (Exercise 5.3.) Suppose that 6-month, 12-month, 18-month, 24-month, and 30-month zero rate are, respectively, 4%, 4.2%, 4.4%, 4.6%, 4.8%, with continuous compounding. Compute the cash price of a bond with a face value of 100 that will mature in 30 months and pay a coupon of 4% per annum semiannually.

Suggested Solutions:

Recall that this is the present value of all future cash flow. The discounted value of each cash flow (in units) is given as follows:

Years	Cash Flow Source	Cash Flow	Discounted Value
0.5	Coupon	2	$+2e^{-0.040\times 0.5}$
1.0	Coupon	2	$+2e^{-0.042\times 1}$
1.5	Coupon	2	$+2e^{-0.044\times 1.5}$
2.0	Coupon	2	$+2e^{-0.046\times 2}$
2.5	Coupon and Principal	102	$+102e^{-0.048\times 2.5}$
Cash Price of Bond (Sum the last column)			$+98.0405.$

Remark: A coupon that pays 4% semiannually is pegged at the **stated principal/face value**. Hence, the bond will pay a **fixed value of** 2 per periods of 6-month.

Exercise 25. (Exercise 5.4.) A 5-year bond with a yield of 7% (continuously compounded) pays an 8% coupon at the end of each year. The principal of $100.

(i) What is the bond's price?

(ii) What is the bond's duration?

(iii) Use the duration to calculate the effect on the bond's price of a 0.2% decrease in its yield.

(iv) Recalculate the bond's price on the basis of a 6.8% per annum yield and verify that the result is in agreement with your answer to (c).

Suggested Solutions:

(i)

Years	Cash Flow Source	Cash Flow	Discounted Value
1	Coupon[10]	8	$+8e^{-0.07\times 1}$
2	Coupon	8	$+8e^{-0.07\times 2}$
3	Coupon	8	$+8e^{-0.07\times 3}$
4	Coupon	8	$+8e^{-0.07\times 4}$
5	Coupon and Principal	108	$+108e^{-0.07\times 5}$
Bond Price (Sum the last column)			$\$103.051.$

(ii)

Years	Cash Flow Source	Cash Flow	Discounted Value	Weighted Discounted Value ($years)
1	Coupon	8	$+8e^{-0.07\times 1}$	$+1\times 8e^{-0.07\times 1}$
2	Coupon	8	$+8e^{-0.07\times 2}$	$+2\times 8e^{-0.07\times 2}$
3	Coupon	8	$+8e^{-0.07\times 3}$	$+3\times 8e^{-0.07\times 3}$
4	Coupon	8	$+8e^{-0.07\times 4}$	$+4\times 8e^{-0.07\times 4}$
5	Coupon and Principal	108	$+108e^{-0.07\times 5}$	$+5\times 108e^{-0.07\times 5}$
Duration (Sum the last column)				$\$445.54$ years.

Hence,

$$\text{Duration} = \frac{\text{Weighted Discounted Value}}{\text{Bond Price}} = \frac{\$445.54 \text{ years}}{\$103.051} = 4.3235 \text{ years.}$$

(iii) Duration relationship in (3):

$$\Delta B \approx -B \cdot D \cdot \Delta y \approx -\$103.051 \times 4.3235 \cdot (-0.2\%) \approx +\$0.891.$$

(iv) Recalculating bond price in (i) with $y = 6.8$ gives

Years	Cash Flow Source	Cash Flow	Discounted Value
1	Coupon	8	$+8e^{-0.068\times 1}$
2	Coupon	8	$+8e^{-0.068\times 2}$
3	Coupon	8	$+8e^{-0.068\times 3}$
4	Coupon	8	$+8e^{-0.068\times 4}$
5	Coupon and Principal	108	$+108e^{-0.068\times 5}$
Bond Price (Sum the last column)			$\$103.947.$

Indeed, we see that

$$\text{(i)} + \text{(iii)} \approx 103.05 + 0.89 \approx 103.94 \approx \text{(iv)}.$$

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[10] See remark in the previous exercise as to why the coupon paid each year is $8.

Exercise 26. (Exercise 5.5.) The cash prices of 6-month and 1-year Treasury bills are \$94.0 and \$89.0, assuming that the principal is at \$100. A 1.5-year Treasury bond that will pay coupons of \$4 every 6 months currently sells for \$94.84. A 2-year Treasury bond that will pay coupons of \$5 every 6 months currently sells for \$97.12. Calculate the 6-month, 1-year, 1.5-year, and 2-year Treasury zero rates.

Comment: Treasury bills (T-bills) do not pay any coupons.
Comment: Assume continuous compounding.

Remark: The original question in the assignment is vague in the sense that the principal associated with the bills is not stated. In practice, if these are not stated, we work with a principal at 100 units (or equivalently, a principal at \$100). Furthermore, if the type of compounding is not stated, always assume continuous discounting.

Suggested Solutions:

(i) For the 6-month zero rate, we observe that 94.0 is the bond price, discounted with the 6-month rate (with continuous compounding). Let r be the zero rate (in p.a.). This implies that

$$100e^{-r\times 1/2} = 94.$$

Hence, we have

$$r = -2\ln\frac{94}{100} \approx 12.4\%.$$

(ii) For the 12-month zero rate, we observe that 89.0 is the bond price, discounted with the 12-month rate (with continuous compounding). Let r be the zero rate (in p.a.). This implies that

$$100e^{-r} = 89.$$

Hence, we have

$$r = -\ln\frac{89}{100} \approx 11.6\%.$$

(iii) For the 1.5-year zero rate, denote that to be r (in p.a.). This is equivalent to solving for r in the following table:

Years	Cash Flow Source	Cash Flow	Discounted Value
0.5	Coupon	4	$+4e^{-0.124\times 0.5}$
1.0	Coupon	4	$+4e^{-0.116\times 1}$
1.5	Coupon and Principal	104	$+104e^{-r\times 1.5}$
Bond Price (Sum the last column)			$+94.84.$

(Note that the spot rates for 6-month and 12-month periods were determined in (i) and (ii).)

Hence, this is equivalent to solving

$$4e^{-0.5\times 0.124} + 4e^{-0.116} + 104e^{-1.5r} = 94.84.$$

Solving this numerically gives

$$r \approx 11.5\%.$$

(iv) For the 2-year zero rate, denote that to be r (in p.a.). This is equivalent to solving for r in the following table:

Years	Cash Flow Source	Cash Flow	Discounted Value
0.5	Coupon	5	$+5e^{-0.124\times 0.5}$
1.0	Coupon	5	$+5e^{-0.116\times 1}$
1.5	Coupon	5	$+5e^{-0.115\times 1.5}$
2.0	Coupon and Principal	105	$+105e^{-r\times 2.0}$
Bond Price (Sum the last column)			$+97.12.$

Hence, this is equivalent to solving

$$5e^{-0.5\times 0.124} + 5e^{-0.116} + 5e^{-1.5\times 0.115} + 105e^{-5r} = 97.12.$$

Solving this numerically gives

$$r \approx 11.3\%.$$

Exercise 27. (Exercise 5.6.) Suppose that risk-free zero interest rates with continuous compounding are as follows:

Maturity (years)	Rate (% per annum)
1	2.0
2	3.0
3	3.7
4	4.2
5	4.5

Calculate the forward interest rates for the second, third, fourth, and fifth years.

Suggested Solutions: Using the forward interest rate formula in (4), we have

Maturity (years)	Rate $r_0(n)$ (% per annum)	Forward Interest Rate $f_0(n-1,n)$ (% per annum)
1	2.0	N.A
2	3.0	$\frac{3.0\times 2-2.0\times 1}{2-1}=4.0,$
3	3.7	$\frac{3.7\times 3-3.0\times 2}{3-2}=5.1,$
4	4.2	$\frac{4.2\times 4-3.7\times 3}{4-3}=5.7,$
5	4.5	$\frac{4.5\times 5-4.2\times 4}{5-4}=5.7.$

Exercise 28. (Exercise 5.7.) Suppose that a company enters into an FRA that is designed to ensure it will receive a fixed rate of 4% on a principal of \$100 million for a 3-month period starting in 3 years (on September 1, 2025). The FRA is an exchange where LIBOR ($= R_M$) is paid and $R_K = 4\%$ is received for the 3-month period from September 1, 2025, to November 30, 2025. All interest rates are expressed with quarterly compounding. If, on September 1, 2025, the 3-month LIBOR proves to be 4.5% for the 3-month period ahead, what would be the cash flow to the company on November 2025 from the FRA?

Suggested Solution:

Remark: The question is a little vague in the sense that it was not clear if the rates quoted were for p.a. rates or for the 3-month period. In this exercise, we will assume that the quoted rates are for a 3-month period. LIBOR is paid and a fixed rate is received implies a short position in FRA. Note that since the simple quarterly compounding only happens once in the period of 3-months, we have that the total payoff is given by

$$\text{Payoff} = 100 \text{ million} \times (\underbrace{4\%}_{\text{Receive fixed rate}} - \underbrace{4.5\%}_{\text{Pay to LIBOR}}) \times \underbrace{1}_{\text{Compounding only happens once}} = -\$0.5 \text{ million} = \boxed{-\$500,000}.$$

6 Discussion 6

Arbitrage-Free Pricing

Recall from the previous discussion on quantifying the **time value of money** through interest rates. To prevent arbitrages, one has to price certain assets/derivatives such that the “profit at maturity” is effectively zero. This is illustrated through different assets/derivatives as follows:

- The arbitrage-free (today’s, $t = 0$) **forward price** on an investment asset with spot price S_0 that provides no income with maturity T is given by

  $$F_0(T) = \underbrace{S_0}_{\text{Spot Price}} \underbrace{e^{rT}}_{\text{Interest Rate Factor}} .$$

  Note that the factor represents either:

  (i) **Cost of financing** the spot purchase of the asset (today), or equivalently,

  (ii) “Growth in value” of the asset due to interest rates.

- The arbitrage-free (today’s, $t = 0$) forward price on an investment asset with spot price S_0 that **provides a known cash income** during the life of the forward contract with the present (discounted) value of I_0 (at $t = 0$) is given by

  $$F_0(T) = S_0 e^{rT} - \underbrace{I_0}_{\text{Present value of income}} \underbrace{e^{rT}}_{\text{Interest Rate Factor}} = (S_0 - I_0)e^{rT}.$$

  Idea: Price it like a forward contract, but due to cash income, this is at a “discounted price” (which makes up for some of the cost of procuring the underlying asset).

- The arbitrage-free (today’s, $t = 0$) forward price on an investment asset with an average **yield** per annum during the life of the forward contract with continuous compounding with spot price S_0 is given by

  $$F_0(T) = S_0 \underbrace{e^{(r-q)T}}_{\text{Interest Rate Factor “discounted” by yield.}}$$

  Idea: Price it like a forward contract, but due to a continuously compounded yield, this acts as a discount factor to the price on a forward contract.

- The **value** of a forward contract at time $t \in [0, T]$ for the long and short position is given by

  $$f_t^{\text{long}} = \underbrace{(F_t(T) - F_0(T))}_{\text{“Profit” at maturity}} \cdot \underbrace{e^{-r(T-t)}}_{\text{Interest Rate Discount Factor (back to time } t \text{ from } T)}$$

  and

  $$f_t^{\text{short}} = (F_0(T) - F_t(T)) \cdot e^{-r(T-t)}$$

  with $F_t(T)$ denoting the arbitrage-free forward price at t. The interest rate r (per annum) is quoted at time t.

  To understand this formula, observe that

  - $$F_t(T)e^{-r(T-t)} = S_t e^{r(T-t)} e^{-r(T-t)} = S_t$$
  - $F_0(T)$ is equivalent to the forward price (“strike price”) of the forward contract at maturity.
  - Hence, the value of the forward contract at time t is equal to

  $$\begin{aligned} f_t^{\text{long}} &= \text{Value of contract today } t \\ &= \text{Profit realized today if the position is “closed” today } t \text{ (that’s how much you can charge for it/it is worth)} \\ &= \underbrace{S_t}_{\text{Buy the asset today}} - \underbrace{F_0(T)}_{\text{Oblidged to sell at this price at maturity}} \underbrace{e^{-r(T-t)}}_{\text{Discount factor from } T \text{ to } t\text{; Equivalent forward price today}} \\ &= F_t(T)e^{-r(T-t)} - F_0(T)e^{-r(T-t)} \\ &= (F_t(T) - F_0(T))e^{-r(T-t)}. \end{aligned}$$

Hence, the value of the contract at time $t = 0$ must be zero, since closing the contract upon time $t = 0$ must yield a profit of 0 (else there will be arbitrage opportunity).

- The arbitrage-free (today's, $t = 0$) forward **exchange rate** with spot price S_0 is given by

  $$F_0(T) = S_0 e^{(r-r_f)T},$$

  where r_f is the interest rate earned on foreign current (which acts as a yield).

  Note: These are measured in the "domestic currency", USD. Furthermore, one can treat the foreign currency as an "asset" (bought with USD).

- The arbitrage-free (today's $t = 0$) forward price of a **commodity** which is an **investment asset** with spot price S_0 is given by

  $$F_0(T) = (S_0 + U_0)e^{rT}$$

  or

  $$F_0(T) = S_0 e^{(r+u)T},$$

  where

  - $U_0 =$ Present value at time $t = 0$ of all storage costs/net income during the life of the forward contract, and
  - u is the storage costs per annum as a proportion of the spot price (net of any yield earned) on the commodity (this is used if the storage cost is proportional to the price of the commodity).
  - $U_0 > 0$ if the storage cost is higher than the income it generates and vice versa.

  Note that for a consumption asset, the arbitrage-free forward price is given by either $F_0(T) \leq (S_0 + U_0)e^{rT}$ or $F_0(T) \leq S_0 e^{(r+u)T}$. This is due to the reluctance to sell it in the spot market and buy it in the forward market. [11]

Note that we will encounter arbitrage strategies forcing us to sell an asset that is not owned. This is known as **short selling**. In principle, we are actually just "borrowing the asset to sell". However, one would have to pay the broker any income that would be received on the asset/securities shorted.

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[11] For instance, if the forward price is lower than expected, in principle, we should expect an arbitrage opportunity. This involves buying it in the forward market for a lower price and selling/shorting it in the spot market. However, due to the reluctance mentioned above, it might not be "possible" to actually do that in practice. (Limited arbitrage.)

Exercise 29. (Exercise 6.1.) Suppose that you enter into a 6-month forward contract on a non-dividend paying stock when the stock price is \$30 and the risk-free interest rate (with continuous compounding) is 5% per annum.

(a) Compute the arbitrage-free forward price.

(b) Show that if the forward price is at \$32 then there exists an arbitrage opportunity. Describe the arbitrage strategy and provide the exact arbitrage profit in 6 months.

(c) Show that if the forward price is \$28 then there exists an arbitrage strategy. Describe the arbitrage opportunity and provide the exact arbitrage profit in 6 months.

Suggested Solutions:

(a) Arbitrage-free forward price (with no income and $T = \frac{1}{2}$ years):

$$F_0(T) = S_0 e^{rT} = \$30 e^{0.05 \times \frac{1}{2}} = \boxed{\$30.76}.$$

(b) Strategy: Buy stock today and enter the short position in the forward contract.

$$\text{Profit at maturity} = \underbrace{\$32}_{\text{Amount obtained by shorting at maturity}} - \underbrace{\$30 e^{0.05 \times \frac{1}{2}}}_{\text{Cost of financing to buy asset at } t = 0} \approx \boxed{\$1.24}.$$

(c) Strategy: Short stock today and enter the long position in the forward contract.

$$\text{Profit at maturity} = \underbrace{\$30 e^{0.05 \times \frac{1}{2}}}_{\text{Amount obtained from shorting, compounded with interest rates}} - \underbrace{\$28}_{\text{Cost of purchasing asset at maturity}} \approx \boxed{\$2.76}.$$

Exercise 30. (Exercise 6.2.) Consider a forward contract to purchase a coupon-bearing bond whose current price is \$900. Suppose that the forward contract matures in 9 months. In addition, suppose that a coupon payment of \$40 is expected on the bond after 4 months. Assume that the 4-month and 9-month risk free interest rates (continuously compounded) are 3% and 4% per annum respectively.

(a) Compute the arbitrage-free forward price.

(b) Show that if the forward price is at \$910, then there exists an arbitrage opportunity. Describe the arbitrage strategy and provide the exact arbitrage profit in 9 months.

(c) Show that if the forward price is at \$870, then there exists an arbitrage strategy. Describe the arbitrage opportunity and provide the exact arbitrage profit in 9 months.

Suggested Solutions:

(a) The present value of all future cash flows is equal to

$$I_0 = \$40e^{-0.03\times\frac{4}{12}} = \$39.602.$$

Hence, the arbitrage-free pricing for the bond with income (with $T = \frac{9}{12}$ years) is given by

$$F_0(T) = \$(900 - 39.602)e^{0.04\times\frac{9}{12}} = \$886.601 \approx \boxed{\$886.60}.$$

(b) Strategy: Borrow money to buy bond today and enter the short position in the forward contract.

$$\text{Profit at maturity} = \underbrace{\$910}_{\text{Amount obtained by shorting the bond at maturity}} - \underbrace{\$886.60}_{\text{Value of bond at maturity is the } t\,=\,0 \text{ Arbitrage-free price}} \approx \boxed{\$23.40}.$$

(Note that you will pay \$900, but due to the interest rates and coupon payment, you will effectively have paid the arbitrage-free price at maturity.)

Remark: Here are some details about the question. Note that the current price of the bond at \$900 implies that this is the current value of all discounted future cash flow. We have computed in (a) that the current value of the coupon is at \$39.602, which means that the value calculated under 900 − 39.602 is the current value of the principal. This in turn implies that the arbitrage-free forward price computed in (a) is the principal stated on the bond.

The detailed version of the arbitrage strategy is as follows:

- Borrow \$39.602 at 3% for 4 months, and
- Borrow \$900 − 39.602 = \$860.398 at 4% for 9 months.
- Now, use this to buy the bond and enter the short position in the forward contract.

What will happen is as follows:

- In 4 months, \$39.602 will grow to \$40, which you will use to pay the first loan that you had. Equivalently, you are using the coupon to pay for your first loan.
- In 9 months (or 5 months from the 4 month-mark), you will obliged to short the bond (before getting the principal). On the other hand, the amount that you have borrowed (at \$860.398) is now at \$886.60, which is precisely the principal stated in the bond. This implies that you get a profit of \$910 − \$886.60, and hence \$23.40 as computed above.

(c) Strategy: Short (issue) bond today (loan the money for interest payments) and enter the long position in the forward contract.

$$\text{Profit at maturity} = \underbrace{\$886.60}_{\text{Value of bond at maturity is the } t\,=\,0 \text{ Arbitrage-free price}} - \underbrace{\$870}_{\text{Amount paid for the bond}} \approx \boxed{\$16.60}.$$

Exercise 31. (Exercise 6.3.) A stock index currently stands at 350. The risk-free interest rate is 4% per annum (with continuous compounding) and the dividend yield on the index is 3% per annum. What should the futures price for a 4-month contract be?

Suggested Solution:

The arbitrage-free future price with yield (with $T = \frac{4}{12}$ years) is given by

$$F_0(T) = S_0 e^{(r-q)T} = \$350 e^{(0.04-0.03)\times\frac{4}{12}} = \boxed{\$351.17}.$$

Exercise 32. (Exercise 6.4.) A 1-year long forward contract on a non-dividend-paying stock is entered into when the stock is \$40 and the risk-free rate of interest is 5% per annum with continuous compounding.

(a) What are the forward price and the initial value of the forward contract?

(b) Six months later, the price of the stock is \$45 and the risk-free interest rate is still 5%. What are the forward price and the value of the forward contract?

Suggested Solutions: Let $T = 1$ (in years).

(a) Arbitrage-free forward price is given by

$$F_0(1) = \$40e^{0.05} = \$42.0508 \approx \boxed{\$42.05}.$$

The initial value of the forward contract is zero. (This can be obtained using the formula $f_t^{\text{long}} = (F_t(1) - F_0(1))e^{-r(1-t)}$ with $t = 0$ or by arguing that an arbitrage-free forward price is obtained such that the value of the contract is zero (else this would result in an arbitrage).)

(b) The forward price is given by

$$F_{0.5}(1) = S_0e^{r(T-t)} = \$45e^{0.05\times(1-0.5)} = \$46.1392 \approx \boxed{\$46.14}.$$

The value of the forward contract (in the long position) is given by

$$f_{0.5}^{\text{long}} = (F_{0.5}(1) - F_0(1))e^{-0.05(1-0.5)} = \$(46.1392 - 42.0508)e^{-0.05(1-0.5)} = \$3.98746 \approx \boxed{\$3.99}.$$

Exercise 33. (Exercise 6.5.) The 2-month interest rates in Switzerland and the United States are 1% and 2% per annum with continuous compounding respectively. The spot price of the Swiss franc is \$1.0500. The futures price for a contract deliverable in 2 months is \$1.0500. What arbitrage opportunities does this create?

Suggested Solutions:

Intuition: Getting the Swiss franc will yield a lower interest rate compared to having the USD (and hence causing one to “lose money” on it). Hence, the idea is to instead “short”/borrow the Swiss franc (so this loss would now be a gain).[12]

Strategy: Borrow the Swiss franc, sell the Swiss franc, and enter a forward contract to return the correct amount of Swiss franc at maturity.

Profit at maturity for each Swiss franc

$$\begin{aligned}
&= \$ \underbrace{1.0500}_{\text{Borrow 1 Swiss Franc, Sell it for USD,}} \underbrace{e^{0.02\times\frac{2}{12}}}_{\text{USD earns this interest rate}} \\
&\quad - \underbrace{1.0500}_{\text{At maturity, buy back the Swiss franc for this price; long position on forward contract.}} \times \underbrace{e^{0.01\times\frac{1}{12}}}_{\text{Amount of Swiss Franc to return for 1 Swiss franc borrowed due to I.R.}} \\
&= \$1.0500(e^{0.02\times\frac{2}{12}} - e^{0.01\times\frac{2}{12}}) \\
&\approx +\$0.001754/\text{Swiss franc.}
\end{aligned}$$

Remark: To use the formula at the start of the discussion supplement, one computes the correct arbitrage-free forward exchange rate today (with $T = \frac{2}{12}$ years) to be

$$F_0(T) = \$1.0500e^{(0.02-0.01)\times\frac{2}{12}} = \$1.05175.$$

As the futures price is not priced at this value (per Swiss franc), this creates an arbitrage opportunity. The corresponding arbitrage opportunity is described above.

Remark: As mentioned above, borrowing the Swiss franc is equivalent to short selling. As stated in the lecture notes/textbook, the short seller has to pay the broker any income that would be received on the Swiss franc “shorted”, and this is equivalent to the interest rates on the Swiss franc!

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[12] Recall that this is equivalent to short selling.

Exercise 34. (Exercise 6.6.) The spot price of silver is \$25 per ounce. The storage costs are \$0.24 per ounce per year, payable quarterly in advance. Assuming that interest rates are 5% per annum for all maturities, calculate the arbitrage-free futures price of silver for delivery in 9 months.

Hint: Use the fact that silver is considered an investment asset and treat the storage costs as negative income.

Suggested Solution: For each quarter, one pays \$0.24/4 = \$0.06 (and this is a fixed for each quarter).

Hence, the current value of all storage income (costs) for silver per ounce is given by

$$U_0 = \$0.06 + \$0.06e^{-0.05\times\frac{3}{12}} + \$0.06e^{-0.05\times\frac{6}{12}} = \$0.177773.$$

The arbitrage-free futures price for silver per ounce is given by

$$F_0(T) = (S_0 + U_0)e^{rT} = \$(25 + 0.177773)e^{0.05\times\frac{9}{12}} \approx \boxed{\$26.14}.$$

7 Discussion 7

Arbitrage-Free Pricing
Extending from the previous discussion, here's a recap of the relevant concepts on the arbitrage-free pricing:

- The arbitrage-free (today's, $t = 0$) forward **exchange rate** with spot price S_0 is given by

  $$F_0(T) = S_0 e^{(r-r_f)T},$$

  where r_f is the interest rate earned on foreign current (which acts as a yield).

  Note: These are measured in the "domestic currency", USD. Furthermore, one can treat the foreign currency as an "asset" (bought with USD).

- The arbitrage-free (today's $t = 0$) forward price of a **commodity** which is an **investment asset** with spot price S_0 is given by

  $$F_0(T) = (S_0 + U_0)e^{rT}$$

  or

  $$F_0(T) = S_0 e^{(r+u)T},$$

  where

  - $U_0 =$ Present value at time $t = 0$ of all storage costs/net income during the life of the forward contract, and
  - u is the storage costs per annum as a proportion of the spot price (net of any yield earned) on the commodity (this is used if the storage cost is proportional to the price of the commodity).
  - $U_0 > 0$ if the storage cost is more than the income it generates and vice versa.

  Note that for a consumption asset, the arbitrage-free forward price is given by either $F_0(T) \leq (S_0 + U_0)e^{rT}$ or $F_0(T) \leq S_0 e^{(r+u)T}$. This is due to the reluctance to sell it in the spot market and buy it in the forward market. [13]

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[13] For instance, if the forward price is lower than expected, in principle, we should expect an arbitrage opportunity. This involves buying it in the forward market for a lower price and selling/shorting it in the spot market. However, due to the reluctance mentioned above, it might not be "possible" to actually do that in practice. (Limited arbitrage.)

Swaps

Definition: A **swap** is an over-the-counter derivatives agreement between two parties to **exchange cash flows** in the future. The agreement defines the dates when the cash flows are to be paid and the way in which they are to be calculated.

Example: A forward contract settled in cash can be seen as an example of a simple swap with only one exchange of cash flow.

Another Example: Interest Rate Swap.

- Idea: "Swap" out one predetermined **fixed rate** (= **swap rate**) on a notional principal for a **floating rate** (mostly LIBOR rates) (or vice versa).
- Each exchange of cash flows is a **forward rate agreement**.
- These are usually traded on electronic platforms and must be cleared through center counterparties (if both parties' main activities are financial) or through an **intermediary** (usually an investment bank). The **intermediary** usually earns a **spread** - this is to compensate the intermediary for its overheads and for potential losses in the event of a default by a counterparty.

Another Example: **Fixed-for-fixed currency swap.**

- Idea: "Swap" out a (fixed) interest rate in one currency for a (fixed) interest rate in another.

In general, the purposes of a swap are to

1. Convert a liability (i.e loan) in one form to another (ie fixed to floating rates; one currency to another currency).
2. Convert an asset in one form to another (ie fixed to floating rates; one currency to another currency).

Both parties in a swap stand to gain from a swap. To perform the correct details for the swap, one would have to employ a **comparative advantage argument**. This allows both parties to benefit only if

- Party A with a comparative advantage for a fixed rate wants to take a loan (have an investment) at a floating rate, and
- Party B with a comparative advantage for a floating rate wants to take a loan (have an investment) at a fixed rate.

This should be done such that it is equally attractive to both companies and accounting for any spread that a bank may take, though many variations do occur in practice. More details on this will be given in the examples below, and the textbook.

Exercise 35. (Exercise 7.1.) Suppose the current EUR/USD exchange rate is \$1.2000 per € (euros). The six-month forward exchange rate is \$1.1950 per €. The six-month USD interest rate is at 1% per annum continuously compounded. Estimate the six-month € interest rate.

Suggested Solutions: Suppose that the six-month forward exchange rate is the arbitrage-free exchange rate. This implies that

$$\$1.1950 = F_0(T) = S_0 e^{(r-r_f)T} = \$1.2000 e^{(0.01-r_f)\times\frac{1}{2}}$$

(with r_f denoting the 6-month euro interest rate) which implies that

$$e^{-\frac{1}{2}r_f} = \frac{1.1950 e^{-0.01\times\frac{1}{2}}}{1.2000}$$

and hence

$$r_f = -2\ln\left(\frac{1.1950 e^{-0.01\times\frac{1}{2}}}{1.2000}\right) = 0.0183507... \approx \boxed{1.84\%}.$$

Exercise 36. (Exercise 7.2.) The spot price of oil is \$50 per barrel and the cost of storing a barrel of oil for one year is \$3, payable at the end of the year. The risk-free interest rate is 5% per annum continuously compounded. What is an upper bound for the one-year futures price of oil?

Suggested Solutions:

Using the formula for a commodity, we see that the arbitrage-free futures price satisfies the following bound: $F_0(T) \leq (S_0 + U_0)e^{rT}$.

The today's discounted value of the storage cost U_0 is given by

$$U_0 = \$3e^{-0.05\times 1} \approx \$2.85369$$

By plugging in the relevant numbers, we have

$$F_0(T) \leq \$(50 + 2.85369)e^{0.05\times 1} \approx \boxed{\$55.56}.$$

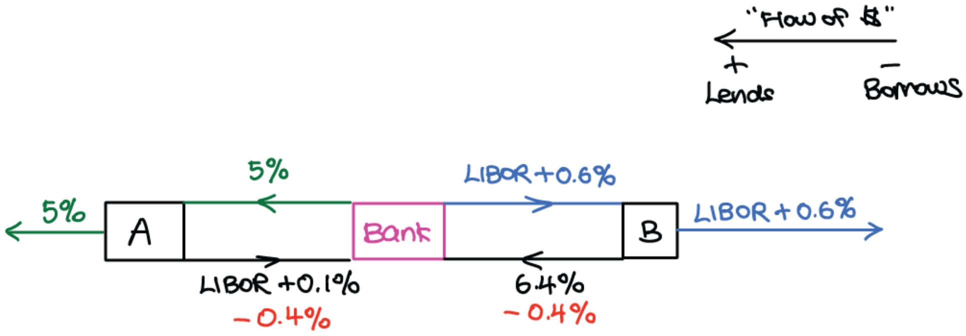
Exercise 37. (Exercise 7.3.) Companies A and B have been offered the following rates per annum on a $20 million five-year **loan**:

	Fixed Rate	Floating Rate
Company A	5.0%	LIBOR + 0.1%
Company B	6.4%	LIBOR + 0.6%

Company A requires a floating-rate loan; Company B requires a fixed-rate loan. Design a swap that will net a back, acting as an intermediary, taking a spread of 0.1% per annum, and will appear equally attractive to both companies.

Suggested Solutions:

1. Identify comparative advantage and compute total gain for all three parties (two companies + bank).
   - Company B pays 1.4 % more at a fixed rate, but only 0.5 % more at a floating rate.
   - Total gain = Difference in "comparative" rates = $1.4\% - 0.5\% = 0.9\%$.
2. Compute the gain for each party.
   - Bank takes a spread of 0.1% per annum. This leaves a total of 0.8% in gain shared between companies A and B.
   - For the swap to be equally attractive, both companies must have a gain of $+0.4\%$ each
3. Draw a swap diagram and describe the relevant details on a swap diagram (ie compute loss, gain, etc). The swap diagram is included below.
   - Each company borrows from others at their comparative advantage rate and lends to the bank at the same rate.
   - Each company then adjusts the "non"-comparative advantage rate in which they would **borrow, down by the gain computed in step 2.** This gets reflected directly into gains for each of the companies.



A : Borrows from others at 5% (p.a.).
Lends bank at 5% (p.a.).
(Gain)
Borrows from bank at LIBOR + 0.1% − 0.4% (p.a.) .

Borrows @ floating rate LIBOR − 0.3%

B : Borrows from others at LIBOR + 0.6%.
Lends bank at LIBOR + 0.6% .
(Gain)
Borrows from bank at fixed rate 6.4 % − 0.4% (p.a.) .

Borrows @ fixed rate 6% .

Bank: Pays LIBOR + 0.6% − (LIBOR + 0.1% − 0.4%) = 0.9%
Receives 1.0% extra fixed I.R

Earns a spread of 0.1% .

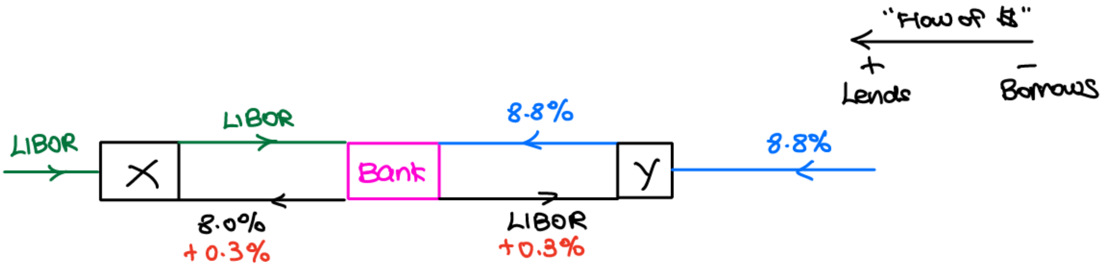
Exercise 38. (Exercise 7.4.) Companies X and Y have been offered the following rates per annum on a $5 million 10-year **investment**:

	Fixed Rate	Floating Rate
Company X	8.0%	LIBOR
Company Y	8.8%	LIBOR

Company X requires a fixed-rate investment; Company Y requires a floating-rate investment. Design a swap that will net a back, acting as an intermediary, taking a spread of 0.2% per annum, and will appear equally attractive to X and Y.

Suggested Solutions:

1. Identify comparative advantage and compute total gain for all three parties (two companies + bank).
    - Company Y pays 0.8 % more at a fixed rate, but at the same floating rate.
    - Total gain = Difference in "comparative" rates = $0.8\% - 0.0\% = 0.8\%$.
2. Compute the gain for each party.
    - Bank takes a spread of 0.2% per annum. This leaves a total of 0.6% in gain shared between companies A and B.
    - For the swap to be equally attractive, both companies must have a gain of $+0.3\%$ each
3. Draw a swap diagram and describe the relevant details on a swap diagram (ie compute loss, gain, etc). The swap diagram is included below.
    - Each company receives (from others) an investment that yields a return on their comparative advantage rate and lends to the bank at the same rate.
    - Each company then adjusts the "non"-comparative advantage rate which they would **receive, up by the gain computed in step 2**. This gets reflected directly into gains for each of the companies.



A : Receives floating rate LIBOR from investment (from others)
Borrows from bank at LIBOR .
Receives from (lend) bank at fixed rate 8.0% + 0.3% (Gain)

Receives 8.3% fixed rate from investment .

B : Receives fixed rate 8.8% from investment (from others)
Borrows from bank at 8.8% ,
Receives from (lend) bank at floating rate LIBOR + 0.3% (Gain)

Receives LIBOR + 0.3% floating rate from investment .

Bank: + LIBOR − (LIBOR + 0.3%)
+ 8.8% − (8.0% + 0.3%) = +0.2%

Earns a spread of 0.2%

Exercise 39. (Exercise 7.5.) Companies X and Y have been offered the following rates per annum on a loan in Japanese Yen ¥ and US Dollar (USD):

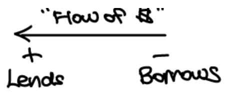
	¥	USD
Company X	5.0%	9.6%
Company Y	6.5%	10.0%

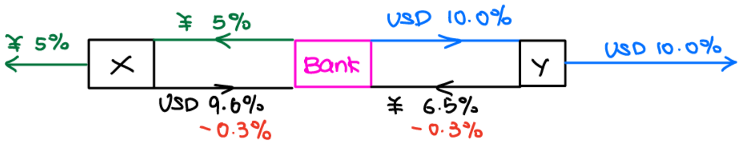
Company X requires a loan in USD; Company Y requires a loan in ¥. Design a swap that will net a back, acting as an intermediary, taking a spread of 50 basis points per annum. Make the swap equally attractive to the two companies and ensure that all foreign exchange risk is assumed by the bank.

Remark: 1 basis point = 0.01%.

Suggested Solutions:

1. Identify comparative advantage and compute total gain for all three parties (two companies + bank).
    - Company Y pays 1.5 % more in ¥, but only 0.4% more in USD.
    - Total gain = Difference in "comparative" rates $= 1.5\% - 0.4\% = 1.1\%$.
2. Compute the gain for each party.
    - Bank takes a spread of 50 basis points $= 0.5\%$ per annum. This leaves a total of 0.6% in gain shared between companies A and B.
    - For the swap to be equally attractive, both companies must have a gain of $+0.3\%$ each
3. Draw a swap diagram and describe the relevant details on a swap diagram (ie compute loss, gain, etc). The swap diagram is included below.
    - Each company lends (from others) at their respective comparative advantage rate and borrows from the bank at the same rate.
    - Each company then adjusts the "non"-comparative advantage rate (the currency in which the companies would like to borrow in) which they would **borrow, down by the gain computed in step 2.** This gets reflected directly into gains for each of the companies.





A : Borrows ¥ from others at 5% (p.a.).
Lends ¥ to bank at 5%
Borrows USD from bank at 9.6% − 0.3% (Gain)

Borrows USD @ 9.3%

B : Borrows USD from others at 10.0%
Lends USD to bank at 10.0%
Borrows ¥ from bank at 6.5% − 0.3% (Gain)

Borrows ¥ @ 6.2%

Bank: − ¥ 5% + ¥ (6.5% − 0.3%)
− USD 10% + USD (9.6% − 0.3%)
= + ¥ 1.2% , − USD 0.7%

Earns a spread of 0.5% (w/ foreign exchange risk)

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8 Discussion 8

Option Markets and Properties of Options.
Recall the following definitions for a European and American option:

- An **American** (call/put) option is one that is **free** to be exercised **any time** before maturity.
- A **European** (call/put) option is one that **can only be** exercised **at** maturity.

Notations:

- $K =$ Strike Price. $(K > 0.)$
- $T =$ Maturity.
- $(S_t)_{0 \leq t \leq T} =$ Spot Price at time t (of the underlying asset).
- $C_t(K,T) =$ (Price) Value of a **European call** option at time $t \in [0,T]$ with strike K and maturity T.
- $P_t(K,T) =$ (Price) Value of a **European put** option at time $t \in [0,T]$ with strike K and maturity T.
- $C_t^{\text{am}}(K,T) =$ (Price) Value of a **American call** option at time $t \in [0,T]$ with strike K and maturity T.
- $P_t^{\text{am}}(K,T) =$ (Price) Value of a **American put** option at time $t \in [0,T]$ with strike K and maturity T.
- $$\begin{aligned} C_0(K,T) &= \text{Value of European Call option at time } t = 0 \\ &= \text{Premium/Price for European Call option (payable at } t = 0). \end{aligned}$$

  A similar description holds for the European/American call/put option with subscript $t = 0$.
- The **intrinsic** value of an option at time t is the payoff at time $t \in [0,T]$ (that is, if the option is to be exercised today if needed to be, what is the profit).
- $$\begin{aligned} C_T(K,T) &= \text{Value of European Call option at time } t = T \\ &= \text{Payoff at time } t = T. \end{aligned}$$

  The second equality is deduced using the definition of value in the previous pointer. A similar description also holds for the European/American call/put option with subscript $t = T$.
- For any $t \in (0,T)$, we have to pick an appropriate arbitrage-free price (value) for $C_t(K,T)$. These prices are covered in the following pages.

Additional/Miscellaneous Notations/Definitions:

- An option is **at-the-money (ATM)** if its intrinsic value (at that time) is zero.
- An option is **in-the-money (ITM)** if its intrinsic value (at that time) is positive.
- An option is **out-of-the-money (OTM)** if its intrinsic value (at that time) is negative.
- The **time value** of an option at time $t \in [0,T]$ is the value of an option arising from the time left to maturity. It is defined as

  $$\text{time value}_t = \text{option price}_t - \text{value}_t.$$

  By definition, one should be able to see that time $\text{value}_T = 0$.

Properties of a European Option:

- Lemma 11.1 - Monotonicity. For all $t \in [0,T]$, if $K_1 \leq K_2$, then

  $$C_t(K_1,T) \geq C_t(K_2,T) \quad \text{and} \quad P_t(K_1,T) \leq P_t(K_2,T).$$

  Intuition: A call option with a higher strike price has a lower payoff function (over all possible S_T), is thus less "powerful", and should thus cost less. A similar argument for a put option holds.

- Lemma 11.1 - Convexity. For all $t \in [0,T]$, the mappings $K \mapsto C_t(K,T)$ and $K \mapsto P_t(K,T)$ are convex. For example,

  $$C_t\left(\frac{K_1+K_2}{2},T\right) \leq \frac{1}{2}C_t(K_1,T) + \frac{1}{2}C_t(K_2,T).$$

- The arbitrage-free **European call option** prices $C_t(K,T)$ on a non-dividend paying stock satisfies:

  $$\boxed{(S_t - Ke^{-r(T-t)})^+ < C_t(K,T) < S_t}$$

  for all $t \in [0,T)$.

- The arbitrage-free **European put option** $P_t(K,T)$prices on a non-dividend paying stock satisfies:

  $$\boxed{(Ke^{-r(T-t)} - S_t)^+ < P_t(K,T) < Ke^{-r(T-t)}}$$

  for all $t \in [0,T)$.

- **Put-Call Parity**. The relationship between arbitrage-free **European call** and **put** option prices written on the same non-dividend paying stock satisfies

  $$\boxed{C_t(K,T) + Ke^{-r(T-t)} = S_t + P_t(K,T)}$$

  for all $t \in [0,T]$.

  Remark: To help you remember this, we will use the **"Law of one price"** mentioned in the lecture notes. At time $t = T$, we have

  $$C_T(K,T) + K = S_T + P_T(K,T).$$

  The left-hand side reads "a call option with strike price K and cash of K". The payoff will thus be given by $\max\{S_T, K\}$ since if $S_T > K$, use the cash to buy the stock to get S_T - else, keep the cash K.

  The right-hand side reads "a put option with strike price K and the stock itself (worth S_T)". The payoff will thus be also given by $\max\{S_T, K\}$ since if $S_T < K$, sell the stock for K - else, keep the stock worth S_T.

  By the "law of one price", we must have that the two "portfolios" (as described in the lecture notes with values given in the put-call parity equation above) must have the same value for all $t < T$. The portfolio would, at $t = 0$,

  - Portfolio 1: A European call option and cash of Ke^{-rT} (in the bank).
  - Portfolio 2: A European put option and the stock.

  and at $t \in [0,T]$,

  - Portfolio 1: A European call option and cash of $Ke^{-r(T-t)}$.
  - Portfolio 2: A European put option and the stock.

  Refer to the lecture notes for more details.

Properties of an American Option:

- Lemma 10.1. For all $t \in [0, T]$, we have

  $$C_t^{\text{am}}(K, T) \geq C_t(K, T) \quad \text{and } P_t^{\text{am}}(K, T) \geq P_t(K, T).$$

  Intuition: American options can be exercised anytime before maturity, while European options have to be exercised at maturity. Since this gives American options more "power", they should be priced higher.

- Lemma 11.7. In fact, for a **non-dividend** paying stock, it is never optimal to exercise an **American call option** before the expiration date. In particular,

  $$C_t^{\text{am}}(K, T) = C_t(K, T)$$

  for all $t \in [0, T]$.

  Intuition: Exercising early forces one to pay the strike price. If we instead hold on to the cash for the strike price, we can earn interest on it up to maturity.

  The proof will be given in Exercise 42. This uses the "law of one price" proof technique.

- The arbitrage-free **American call option** prices $C_t^{\text{am}}(K, T)$ on a non-dividend paying stock satisfies:

  $$\boxed{(S_t - Ke^{-r(T-t)})^+ < C_t^{\text{am}}(K, T) < S_t}$$

  for all $t \in [0, T)$.

  Remark: This is the same as that for European call options since $C_t^{\text{am}}(K, T) = C_t(K, T)$ by Lemma 11.7.

- The arbitrage-free **American put option** $P_t^{\text{am}}(K, T)$ prices on a non-dividend paying stock satisfies:

  $$\boxed{(K - S_t)^+ < P_t^{\text{am}}(K, T) < K}$$

  for all $t \in [0, T)$.

  Remark: The difference between this formula and that for the European put option is that we have replaced $Ke^{-r(T-t)}$ with K.

- Lemma 11.10 **Put-Call Parity**. The relationship between arbitrage-free **American call** and **put** option prices written on the same non-dividend paying stock is encapsulated in the following inequality:

  $$\boxed{S_t - K \leq C_t^{\text{am}}(K, T) - P_t^{\text{am}}(K, T) \leq S_t - Ke^{-r(T-t)}}$$

  for all $t \in [0, T]$.

Exercise 40. (Exercise 8.1.) What are the upper and lower bounds for the (arbitrage-free) price of a 6-month European call option on a non-dividend-paying stock when the stock price is \$80, the strike price is \$75, and the risk-free interest rate is 10% per annum (continuously compounded)?

Suggested Solutions: Recall that the arbitrage-free bounds for a European call option prices on a non-dividend paying stock is given by

$$(S_t - Ke^{-r(T-t)})^+ < C_t(K,T) < S_t.$$

At time $t = 0$, this reduces to

$$(S_0 - Ke^{-rT})^+ < C_0(K,T) < S_0.$$

From the question, we have $S_0 = 80$, $K = 75$, $r = 10\%$, and $T = \frac{1}{2}$ year. Plugging these in, we have

$$(80 - 75e^{-0.1\times\frac{1}{2}})^+ < C_0(K,T) < 80$$

which simplifies to (with 2 decimal places and units)

$$\boxed{\$8.66 < C_0(K,T) < \$80}.$$

Exercise 41. (Exercise 8.2.) What are the upper and lower bounds for the (arbitrage-free) price of a 2-month European put option on a non-dividend-paying stock when the stock price is \$58, the strike price is \$65, and the risk-free interest rate is 5% per annum (continuously compounded)?

Suggested Solutions: Recall that the arbitrage-free bounds for a European put option prices on a non-dividend paying stock is given by

$$(Ke^{-r(T-t)} - S_t)^+ < P_t(K,T) < Ke^{-r(T-t)}.$$

At time $t = 0$, this reduces to

$$(Ke^{-rT} - S_0)^+ < P_0(K,T) < Ke^{-rT}.$$

From the question, we have $S_0 = 58$, $K = 65$, $r = 5\%$, and $T = \frac{2}{12}$ year. Plugging these in, we have

$$(65e^{-0.05\times\frac{1}{6}} - 58)^+ < P_0(K,T) < 65e^{-0.05\times\frac{1}{6}}$$

which simplifies to (with 2 decimal places and units)

$$\boxed{\$6.46 < P_0(K,T) < \$64.46}.$$

Exercise 42. (Exercise 8.3.) Argue that for a non-dividend paying stock, it is never optimal to exercise an American call option before the expiration date. In particular, we have

$$C_t^{\text{am}}(K,T) = C_t(K,T)$$

for all $t \in [0,T]$.

Suggested Solutions:

By the put-call parity, we have

$$C_t(K,T) + Ke^{-r(T-t)} = S_t + P_t(K,T)$$

This implies that

$$C_t(K,T) = S_t + P_t(K,T) - Ke^{-r(T-t)} > S_t - K$$

for $t < T$ (since $P_t(K,T) \geq 0$).

By Lemma 10.1, since $C_t^{\text{am}}(K,T) \geq C_t(K,T)$, we then have

$$C_t^{\text{am}}(K,T) > S_t - K.$$

Observe that $S_t - K$ represents the payoff obtained by exercising the American call option at time t (if $S_t > K$), which is always less than what the option itself is worth (ie the value of the American call option at time t).

Remark: Why is this not true for an American put option? Recall that the payoff of a European put option is given by $(K - S_T)^+$. Due to the negative sign in front of S_T and the fact that S_T can never fall below 0, at $S_T = 0$, it is better to exercise the American put option to get K and get interest on it. This is because if we don't do so, we are guaranteed to **not make any profit** for the next instance in time (since S_T must either remain at 0 or go up, causing the value of the option to remain or fall).

Exercise 43. (Exercise 8.4.) The price of a non-dividend-paying stock is $19 and the price of a 3-month European call option on the stock with a strike price of $20 is $1. The risk-free rate is 4% per annum (with continuous compounding). What is the price of a 3-month European put option (written on that stock) with a strike price of $20?

Suggested Solution: Since we know the price of a European call option and want to compute the price of a European put option, we turn to the **put-call parity**. This is given by

$$C_t(K,T) + Ke^{-r(T-t)} = S_t + P_t(K,T)$$

Rearranging and setting $t = 0$, we have

$$P_0(K,T) = C_0(K,T) + Ke^{-rT} - S_0.$$

Since $C_0(K,T) = 1$, $K = 20$, $S_0 = 19$, $r = 0.04$, and $T = \frac{3}{12}$ years, we have

$$P_0(K,T) = 1 + 20e^{-0.04\times\frac{3}{12}} - 19 = \boxed{\$1.80}.$$

Exercise 44. (Exercise 8.5.) Suppose that the price of a stock today is at \$25. For a strike price of $K =$ \$24, a 3-month European call option on that stock is quoted with a price of \$2, and a 3-month European put option on the same stock is quoted at \$1.50. Assume that the risk-free rate is 10% per annum.

(i) Does the put-call parity hold?

(ii) Is there an arbitrage opportunity? If yes, explain how the arbitrage opportunity would look like.

Suggested Solutions:

(i) First, we would like to compute the arbitrage-free European call option price given the European put option price. This is given by

$$C_0(K,T) = S_0 + P_0(K,T) - Ke^{-rT}.$$

By plugging in the relevant numbers, we have

$$C_0(K,T) = 25 + 1.50 - 24e^{-0.1\times\frac{3}{12}} \approx \$3.09.$$

However, the cited 3-month European call option is only at \$2. This implies that the European call option is priced **too low**. Hence, the put-call parity does not hold.

(ii) Since the put-call parity does not hold, there is an arbitrage opportunity. The arbitrage strategy at $t = 0$ is as follows:

- (Long) Portfolio 1: Buy (long) the European call option, and set aside cash of Ke^{-rT} to put it into a bank.
- (Short) Portfolio 2: Short the European put option, and short (sell) the stock for S_0.
- In this scenario, since the European call option is priced too low, we have

$$C_0(K,T) < S_0 + P_0(K,T) - Ke^{-rT}. \tag{5}$$

- We would have gained the following amount in cash:

$$B := -C_0(K,T) + P_0(K,T) + S_0 - Ke^{-rT} > 0$$

  by virtue of (5). Put this cash into a bank to earn an interest rate.

At time $t = T$,

- The cash of Ke^{-rT} set aside now grows into K. At this point, we have $Be^{rT} > 0$ in the bank, constituting positive profit.
- Suppose we can show that the K cash from portfolio 1 at maturity, together with the long European call option, the short European put option, and the impacts of short selling a stock, guarantees a non-negative profit. In that case, we are done (our profit is then at least Be^{rT} in this entire strategy).
- If $S_T < K$, we do not exercise the call option. However, the other party with the put option that we have shorted will choose to exercise the put option. We are forced to use the K cash to buy the stock of S_T. We now have a stock, which we would “return” since we did a short selling at $t = 0$.
- If $S_T > K$, we exercise the call option to buy the stock for K. On the other hand, the other party will choose to not exercise their put option. Overall, we will swap K out for the stock. We now have a stock, which we would “return” since we did a short selling at $t = 0$.

In conclusion, the net profit is given by

$$Be^{rT} = (-C_0(K,T) + P_0(K,T) + S_0 - Ke^{-rT})e^{rT} > 0.$$

Remark: This is essentially motivated from the proof of put-call parity, by arguing that the two portfolios must be of the same price for all $t < T$. Here, we are essentially buying the cheaper portfolio (since the European call options are priced too low), and selling the more expensive portfolio.

Exercise 45. (Exercise 8.6.) Calls were traded on exchanges before puts. During the period of time when calls were traded but puts were not traded, how would you create a European put option on a non-dividend-paying stock synthetically? (In other words, explain how to create a financial asset that always has the same value as a put option.)

Suggested Solutions: Recall that in the proof of put-call parity, we have that the following portfolios have the same value at time $t = 0$:

- Portfolio 1: A European call option and cash of Ke^{-rT}.
- Portfolio 2: A European put option and the stock.

Next, we remove (by means of owing if needed) a stock from each portfolio. The new portfolios (still of the same value) are given by

- Portfolio 1: A European call option, cash of Ke^{-rT} (and put the cash in the bank), and owes a third party a stock.
- Portfolio 2: A European put option.

Hence, at any $t \in [0, T]$, the portfolio becomes

- Portfolio 1: A European call option, cash of $Ke^{-r(T-t)}$, and owes a third party a stock.
- Portfolio 2: A European put option.

To show that they have the same value, we invoke the put-call parity formula to obtain

$$C_t(K,T) + Ke^{-r(T-t)} = S_t + P_t(K,T).$$

By rearranging, we have

$$\underbrace{C_t(K,T) + Ke^{-r(T-t)} - S_t}_{\text{Value of Portfolio 1}} = \underbrace{P_t(K,T)}_{\text{Value of Portfolio 2}}.$$

Exercise 46. (Exercise 8.7.) The prices of European call and put options on a non-dividend-paying stock with an expiration date in 12 months and a strike price of $120 are $20 and $5, respectively. The current stock price is $130. What is the implied risk-free rate?

Suggested Solution: Using the put-call parity, we have

$$C_t(K,T) + Ke^{-r(T-t)} = S_t + P_t(K,T).$$

At $t = 0$, we have

$$C_0(K,T) + Ke^{-rT} = S_0 + P_0(K,T).$$

This implies that

$$e^{-rT} = \frac{S_0 + P_0(K,T) - C_0(K,T)}{K}$$

and thus

$$r = -\frac{1}{T}\ln\left(\frac{S_0 + P_0(K,T) - C_0(K,T)}{K}\right)$$

Plugging in the relevant values, we have

$$r = -\frac{1}{1}\ln\left(\frac{130+5-20}{120}\right) = 0.0425596 \approx \boxed{4.26\%}.$$

9 Discussion 9

Binomial Tree Model (Cox-Ross-Rubinstein Model).
Recall the following definitions for a European and American option:

- An **American** (call/put) option is one that is **free** to be exercised **any time** before maturity.
- A **European** (call/put) option is one that **can only be** exercised **at** maturity.

One-step Binomial Model.
Notations:

- T = Maturity (in years).
- r = Risk-free interest rate p.a. (continuously compounded).
- S_0 = Stock price at $t = 0$.
- S_T = Stock price at maturity T, which only takes
  - $S^u := S_0 \cdot u$ = one-step price upward move.
  - $S^d := S_0 \cdot d$ = one-step price downward move.
  - u = increment factor ($u - 1$ representing percentage increase).
  - d = decrement factor ($1 - d$ representing percentage decrease).
- f_0 = Option price at $t = 0$.
- f_T = Option payoff at maturity T, taking only two values:
  - f^u = Option's payoff if stock price moved up.
  - f^d = Option's payoff if stock price moved down.

Replication (No-arbitrage) Arguments

- Idea:
  - Law of One Price
  - Portfolio 1: (European) Option.
  - Portfolio 2: Replicating portfolio with Δ_0 shares/stocks and cash of $V_0 - \Delta_0 S_0$, with total value at $t = 0$ given by V_0.
  - No arbitrage implies
  $$V_0 = \text{Price of Option.}$$
  - Two cases: Stock price goes up or down.
  - Two unknowns, V_0 and S_0.
  - Governing Equation:
  $$\begin{cases} (V_0 - \Delta_0 S_0)e^{rT} + \Delta_0 \underbrace{S_0 u}_{S_T = S^u} & = f^u \\ (V_0 - \Delta_0 S_0)e^{rT} + \Delta_0 \underbrace{S_0 d}_{S_T = S^d} & = f^d \end{cases} \tag{6}$$

- Theorem 13.3 (Part II, Modified). The one-step binomial tree model is **complete**, i.e. every contingent claim f_T is perfectly **replicable** (**attainable**) by the replication strategy
$$\boxed{\Delta_0 = \frac{f^u - f^d}{S^u - S^d}}$$
with the unique arbitrage-free price of the option $f_0 = V_0$ given by
$$\boxed{f_0 = V_0 = \begin{cases} \Delta_0(S_0 - S^u e^{-rT}) + f^u e^{-rT} \\ \Delta_0(S_0 - S^d e^{-rT}) + f^d e^{-rT} \end{cases}}$$
(Here, any of the above expressions for V_0 will work.)

- Δ_0 and V_0 derived by solving the simultaneous equations in (6).

- Since f^u and f^d are just the payoff of any European-style options, the above theorem holds as long as we have the payoff of the form $f_T = h(S_T)$. For an example, see Exercise 49.

- Understanding the exact contents of the two portfolios allows us to construct the correct arbitrage strategy should the options be priced incorrectly. (A similar idea to put-call parity in the previous discussion.)

- $\Delta_0 > 0$ is equivalent to buying (long) the shares/stocks, $\Delta_0 < 0$ is equivalent to short selling.

Risk-Neutral Valuation Arguments

- Assuming that p^* is the probability that the share/stock price goes up.
- Take $p^* \times$ the first equation in (6) and add it to $(1-p^*)\times$ the second equation in (6). We get

$$V_0 + \Delta_0 \underbrace{\left(e^{-rT}(S_0 \cdot u \cdot p^* + S_0 \cdot d \cdot (1-p^*)) - S_0\right)}_{\text{Set }=0} = \underbrace{e^{-rT}(f^u p^* + f^d(1-p^*))}_{\mathbb{E}^*(e^{-rT}F_T)}. \tag{7}$$

- Setting $\left(e^{-rT}(S_0 \cdot u \cdot p^* + S_0 \cdot d \cdot (1-p^*)) - S_0\right) = 0$ is equivalent to saying that

$$S_0 e^{rT} = S_0 \cdot u \cdot p^* + S_0 \cdot d \cdot (1-p^*) = \mathbb{E}^*(S_T).$$

  Equivalently, this choice of p^* implies that the expected value of the stock at maturity is equivalent to putting an equivalent amount of cash in the bank with the risk-free interest rate r.

  Hence, $p^* \in (0,1)$ is known as the **risk-neutral probability**.

- In turn, (7) yields

$$f_0 = V_0 = \mathbb{E}^*(e^{-rT} f_T),$$

  which says that based on our choice of risk-neutral probability p^*, **the option's price is the expected value of the discounted value of the option's payoff.**

- Theorem 13.3 (Part I). The one-step binomial tree model is **free of arbitrage** if and only if the parameters u, d, r satisfy

$$d < e^{rT} < u.$$

  In this case, by picking the **risk-neutral probability** to be

$$\boxed{p^* = \frac{e^{rT}-d}{u-d}} \in (0,1),$$

  the **unique arbitrage-free price** of an option with payoff f_T is given by

$$\boxed{f_0 = \mathbb{E}^*(e^{-rT} f_T) = e^{-rT}\left(p^* \cdot f^u + (1-p^*) \cdot f^d\right)}.$$

- The above principle of pricing derivative is called the **risk-neutral valuation**, where we've introduced an artificial probability of an up movement on the stock price.

Remark: $\mathbb{E}$ refers to the expectation of a random variable. In this case, we are modeling f_T as a random variable that takes on two values, f^u with probability p^* and f^d with probability $1-p^*$. Hence, the expectation taken with respect to this binomial distribution is denoted by $\mathbb{E}^*$, with p^* as the parameter associated with the binomial distribution.

Two-step Binomial Model. Following a similar argument, we have:

Replication (No-arbitrage) Arguments

(Additional) Notation:

- V_0 = Initial value of portfolio.
- Δ_0 = Number of shares to hold at time 0.
- Δ_1^u = Number of shares to hold at time 1 if stock price went up for the first step.
- Δ_1^d = Number of shares to hold at time 1 if stock price went down for the first step.
- f^{uu} = Option's payoff if stock price moved up twice.
- $f^{ud} = f^{du}$ = Option's payoff if stock price moved up once, and down once.
- f^{dd} = Option's payoff if stock price moved down twice.
- $\Delta t = \frac{T}{2}$ = Time elapsed for one time step.

Properties:

- Replication Strategy involves **dynamic hedging** - readjusting the portfolio after the first step (depending on if the stock price went up or down).
- Idea:
  - Four unknowns: $V_0, \Delta_0, \Delta_1^u, \Delta_1^d$.
  - Four equations:

$$\begin{cases} \left((V_0 - \Delta_0 S_0)e^{r\Delta t} + \Delta_0 S_0 u - \Delta_1^u S_0 u\right) e^{r\Delta t} + \Delta_1^u S_0 u^2 & = f^{uu}. \\ \left((V_0 - \Delta_0 S_0)e^{r\Delta t} + \Delta_0 S_0 u - \Delta_1^u S_0 u\right) e^{r\Delta t} + \Delta_1^u S_0 ud & = f^{ud}. \\ \left((V_0 - \Delta_0 S_0)e^{r\Delta t} + \Delta_0 S_0 d - \Delta_1^u S_0 d\right) e^{r\Delta t} + \Delta_1^d S_0 du & = f^{du}. \\ \left((V_0 - \Delta_0 S_0)e^{r\Delta t} + \Delta_0 S_0 d - \Delta_1^u S_0 d\right) e^{r\Delta t} + \Delta_1^d S_0 d^2 & = f^{dd}. \end{cases} \tag{8}$$

- Theorem 13.5 (Part II, Modified). The two-step binomial tree is **complete**, i.e., every contingent claim f_T is perfectly **replicable** (**attainable**) by the **dynamic** replication strategy:

$$\boxed{\Delta_0 = \frac{f^u - f^d}{S_0 u - S_0 d}, \quad \Delta_1^u = \frac{f^{uu} - f^{ud}}{S_0 u^2 - S_0 ud}, \quad \Delta_1^d = \frac{f^{du} - f^{dd}}{S_0 du - S_0 d^2}}.$$

  The value of the initial portfolio can be computed by plugging in the values of Δ_0, Δ_1^u, and/or Δ_1^d into any one of the four equations in (8).

Risk-Neutral Valuation Arguments

- The idea is the same as for the one-step model, with the same risk-neutral probability p^* for both steps and similar interpretations for each of the intermediate steps.
- We assume that the percentage/factor increase/decrease at each time step is the same, with the same risk-neutral probability p^*.
- Theorem 13.5 (Part I). The two-step binomial tree model is **free of arbitrage** if and only if the parameters u, d, r satisfy $d < e^{r\Delta t} < u$. In this case, the risk-neutral probability is given by

$$\boxed{p^* = \frac{e^{r\Delta t} - d}{u - d}},$$

  with the **unique arbitrage-free price** of an option with payoff f_T given by

$$\boxed{f_0 = \mathbb{E}^*(e^{-rT} f_T) = e^{-r\cdot(2\Delta t)} \left((p^*)^2 \cdot f^{uu} + 2 \cdot p^* \cdot (1 - p^*) \cdot f^{ud} + (1 - p^*)^2 \cdot f^{dd}\right)}.$$

Remark: For American-style options, one would have to check at each node prior to maturity to see if early exercise is optimal. This will be explained in detail in the Exercises below - see Exercise 50 and 51.

Exercise 47. (Exercise 9.1.) A stock price is currently \$50. It is known that at the end of 6 months, it will be either \$60 or \$42. The risk-free rate of interest with continuous compounding is at 12 % per annum. Calculate the value of a 6-month European call option on a stock with an exercise price of \$48. Verify that no-arbitrage arguments (i.e. the **replication arguments**) and the **risk-neutral valuation arguments** give the same answer.

Suggested Solutions: Note that this corresponds to a one-step binomial tree model.

Replication Arguments:

First, we compute the payoff given that the stock has increased/decreased in price. Here, we have

$$f^u = (S_T - K)^+ = (60 - 48)^+ = 12$$

and

$$f^d = (S_T - K)^+ = (42 - 48)^+ = 0.$$

Hence, by Theorem 13.3, since $S^u = 60$ and $S^d = 42$, we have

$$\Delta_0 = \frac{f^u - f^d}{S^u - S^d} = \frac{12 - 0}{60 - 42} = \frac{12}{18} = \frac{2}{3}.$$

Hence, the initial value of the portfolio is given by

$$V_0 = \Delta_0(S_0 - S^u e^{-rT}) + f^u e^{-rT} = \frac{2}{3}(50 - 60e^{-0.12\times\frac{1}{2}}) + 12e^{-0.12\times\frac{1}{2}} = 6.96393.$$

This implies that the European call option should be priced at $f_0 = V_0 \approx \$6.96$.

Remark: The alternative expression for V_0 should give the same answer, that is,

$$V_0 = \Delta_0(S_0 - S^d e^{-rT}) + f^d e^{-rT} = \frac{2}{3}(50 - 42e^{-0.12\times\frac{1}{2}}) + 0e^{-0.12\times\frac{1}{2}} \approx 6.96.$$

Risk-Neutral Valuation Arguments:

By Theorem 13.3, the risk-neutral probability is given by

$$p^* = \frac{e^{rT} - d}{u - d}.$$

To utilize this formula, we first compute u and d, given by

$$u = \frac{S^u}{S_0} = \frac{60}{50} = \frac{6}{5}$$

and

$$d = \frac{S^d}{S_0} = \frac{42}{50} = \frac{21}{25}.$$

Hence,

$$p^* = \frac{e^{0.12\times\frac{1}{2}} - \frac{21}{25}}{\frac{6}{5} - \frac{21}{25}} = 0.616213.$$

This implies that we can compute the value of the option at $t = 0$ by

$$f_0 = \mathbb{E}^*(e^{-rT} f_T) = e^{-rT}\left(p^* f^u + (1 - p^*) f^d\right) = e^{-0.12\times\frac{1}{2}}\left(0.616213 \times 12 + (1 - 0.616213) \times 0\right) \quad \approx 6.96.$$

Hence, both arguments give the same price for the European call option.

Exercise 48. (Exercise 9.2.) A stock price is currently \$100. Over each of the next two 6-month periods, it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% p.a. with continuous compounding. Use a two-step binomial tree model and the **risk-neutral valuation approach** to compute

(i) the value of a 1-year at-the-money European call option written on the stock;

(ii) the value of a 1-year at-the-money European put option written on the stock.

Verify that the European call and European put price satisfy put-call parity from Lemma 11.5, Lecture 20.

Remark: Recall that an option is said to be **at-the-money** if the strike price is equal to the current spot price (ie the intrinsic value at $t = 0$ is zero, that is, the payoff if you are allowed to exercise the option now). Hence, $K = 100$ in this exercise.

Suggested Solutions: As mentioned, this constitutes a two-step binomial model, with $u = 1.1$ and $d = 0.9$. For a risk-neutral valuation approach, by Theorem 11.5, we first compute the risk-neutral probability given by

$$p^* = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.08\times\frac{1}{2}} - 0.9}{1.1 - 0.9} = 0.704054.$$

(i) For the European call option, we then compute all possible payoffs corresponding to whether the stock price moved up twice, moved up once and down once, and moved down twice. These are given by

- $f^{uu} = (S^{uu} - K)^+ = (100 \times 1.1^2 - 100)^+ = 21.$
- $f^{ud} = f^{du} = (S^{ud} - K)^+ = (100 \times 1.1 \times 0.9 - 100)^+ = (99 - 100)^+ = 0.$
- $f^{dd} = (S^{dd} - K)^+ = (100 \times 0.9^2 - 100)^+ = 0.$

Hence, the value of the option is given by

$$\begin{aligned} C_0(K,T) = f_0 = \mathbb{E}^*(e^{-rT} f_T) &= e^{-rT}\left((p^*)^2 f^{uu} + 2\cdot p^* \cdot (1-p^*) f^{ud} + (1-p^*)^2 f^{dd}\right) \\ &= e^{-0.08\times 1}(0.704504^2 \times 21 + 2(0.704504)(1-0.704504)\times 0 + (1-0.704504)^2 \times 0) \\ &= \boxed{\$9.60916}. \end{aligned}$$

Remark: $T = 1$ here, representing the time to maturity, which happens after 2 × 6-month periods.

(ii) For the European put option, we have

- $f^{uu} = (K - S^{uu})^+ = (100 - 100 \times 1.1^2)^+ = 0.$
- $f^{ud} = f^{du} = (K - S^{ud})^+ = (100 - 100 \times 1.1 \times 0.9)^+ = (100 - 99)^+ = 1.$
- $f^{dd} = (K - S^{dd})^+ = (100 - 100 \times 0.9^2)^+ = 19.$

Hence, the value of the option is given by

$$\begin{aligned} P_0(K,T) = f_0 = \mathbb{E}^*(e^{-rT} f_T) &= e^{-rT}\left((p^*)^2 f^{uu} + 2\cdot p^* \cdot (1-p^*) f^{ud} + (1-p^*)^2 f^{dd}\right) \\ &= e^{-0.08\times 1}(0.704504^2 \times 0 + 2(0.704504)(1-0.704504)\times 1 + (1-0.704504)^2 \times 19) \\ &= \boxed{\$1.91583}. \end{aligned}$$

To verify that put-call parity hold, we have to check that

$$C_t(K,T) + Ke^{-r(T-t)} = S_t + P_t(K,T)$$

holds at $t = 0$, that is

$$C_0(K,T) + Ke^{-rT} = S_0 + P_0(K,T).$$

Indeed, we have

- LHS $= C_0(K,T) + Ke^{-rT} = 9.60916 + 100e^{-0.08\times 1} = 101.921.$
- RHS $= S_0 + P_0(K,T) = 100 + 1.91583 = 101.916.$

These two values are close, up to rounding errors! Hence, put-call parity is satisfied!

Exercise 49. (Exercise 9.3.) A stock price is currently \$25. Suppose that at the end of 2 months, it will be either \$23 or \$27. The risk-free interest rate is 10% per annum with continuous compounding. Suppose S_T is the stock price at the end of 2 months. What is the value of a derivative that pays off S_T^2 (at $t = 0$)?

Suggested Solution: This corresponds to a one-step binomial model. Note that either the replication argument or the risk-neutral valuation argument would work here. For simplicity, we will present the replication argument.

- First, we compute the payoffs at maturity. These are given by

$$f^u = (S_T)^2 = (S^u)^2 = 27^2 = 729$$

  and

$$f^d = (S_T)^2 = (S^d)^2 = 23^2 = 529.$$

- Next, we compute the delta in the hedging strategy, given by

$$\Delta_0 = \frac{f^u - f^d}{S^u - S^d} = \frac{729 - 529}{27 - 23} = 50.$$

- Last but not least, we compute the price (premium) of the option, which is equivalent to the value of the initial portfolio. This is thus given by

$$f_0 = V_0 = \Delta_0(S_0 - S^u e^{-rT}) + f^u e^{-rT} = 50(25 - 27e^{-0.1\times\frac{2}{12}}) + 729e^{-0.1\times\frac{2}{12}} \approx 639.26.$$

  Hence, the option should be priced at $V_0 = f_0 \approx \boxed{\$639.26}$.

Remark: This is an example of an exotic option known as the (European) **power option** with a strike price of 0.

Exercise 50. (Exercise 9.5, Modified.) A stock price is currently \$40. Over each of the next two 3-month periods, it is expected to go up by 10% or down by 10%. The risk-free interest rate is 12% per annum with continuous compounding.

(i) What is the value of a 6-month European put option with strike price \$42?

(ii) What is the value of a 6-month American put option with strike price \$42?

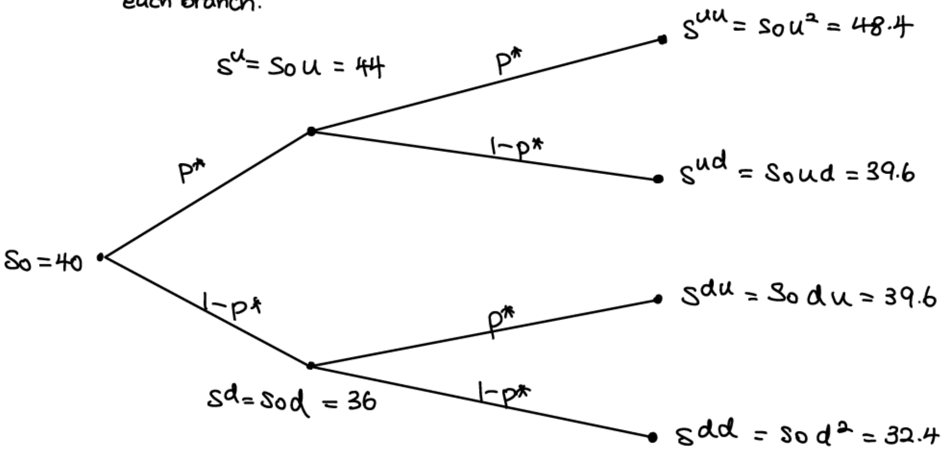
Remark: As an exercise, try to repeat (i) and (ii) in the case of a call option with strike price \$42.

Suggested Solutions: For simplicity, we will employ the risk-neutral valuation argument. Regardless of the type of option, the risk-neutral probability can be computed (with $u = 1.1$ and $d = 0.9$) by

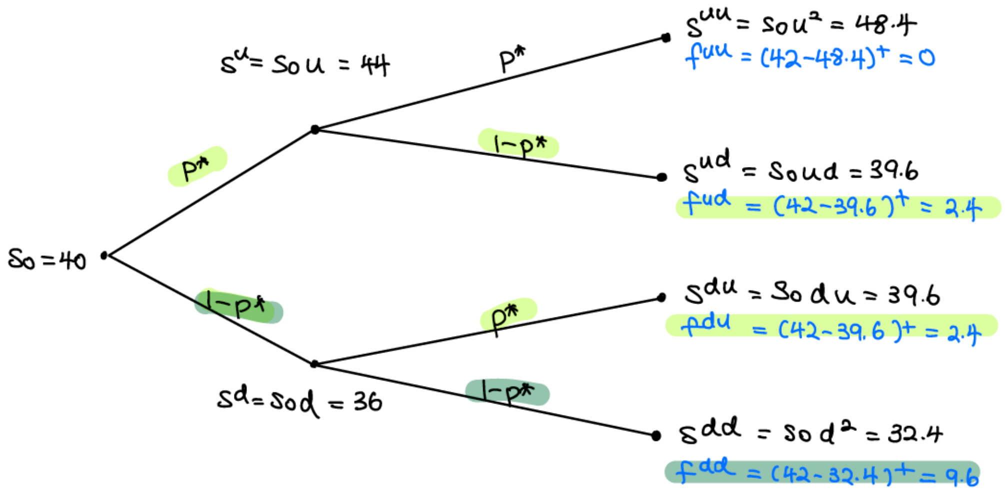
$$p^* = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.12 \times \frac{3}{12}} - 0.9}{1.1 - 0.9} = 0.652273.$$

For pricing an American option, one would have to decide at each node of the binomial tree if early exercise is optimal. To illustrate the idea behind pricing the European option using the risk-neutral valuation argument, we will combine both parts in a binomial tree diagram below.

Step 1. Populate the binomial tree with spot prices and the probability along each branch.



(i) Step 2: For a European option, it can only be exercised at maturity.
Hence, it suffices to compute the payoff at maturity.
Put option payoff $= (K - S_T)^+$, $K = 42$ $(42 - 48.4)^+ = 0$



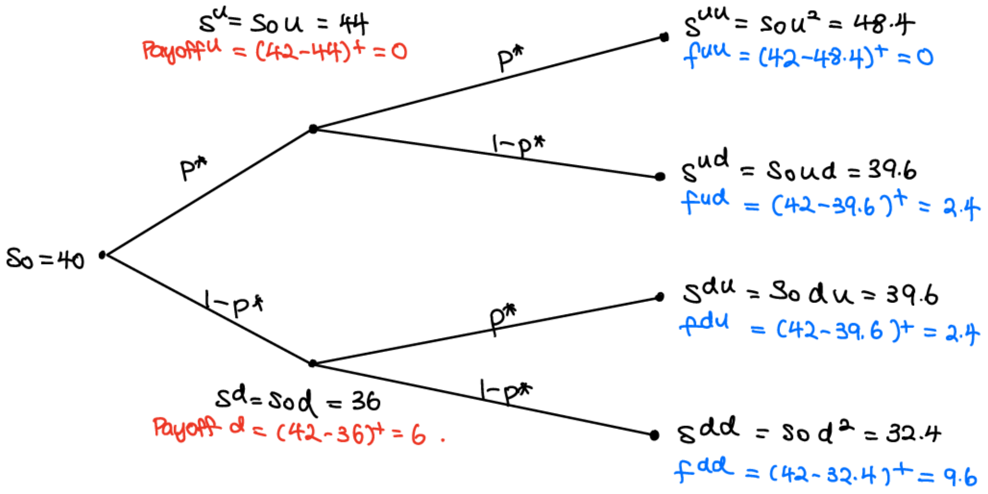
Step 3: Compute the price of the option, given by the expected discounted value of all possible payoffs. $f_0 = \mathbb{E}^*(e^{-rT}f_T)$.

$$f_0 = \mathbb{E}^*(e^{-rT}f_T) = e^{-0.12\times \overset{2\times 3\text{ months}}{\overbrace{\tfrac{6}{12}}}} \times \left(\overset{2\times p^* \times (1-p^*) \times f^{ud}}{2\times 0.652273 \times (1-0.652273) \times 2.4} + \overset{(1-p^*)^2\times f^{dd}}{(1-0.652273)^2\times 9.6} \right)$$

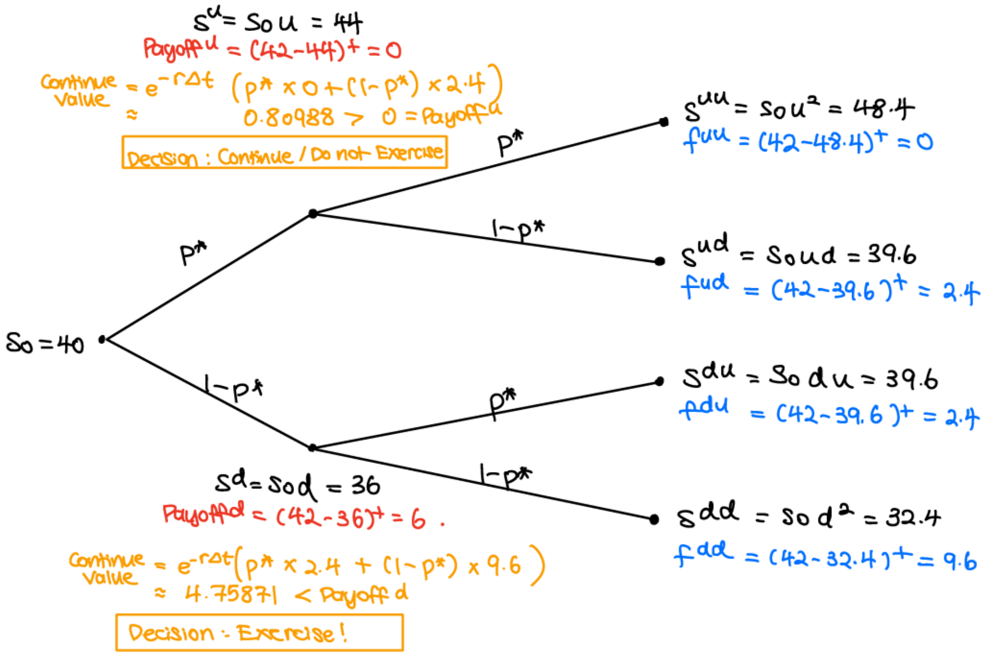
$$\approx \boxed{\$2.12}$$

(ii) Step 2: For an American option, it can be exercised at anytime up to maturity. Hence, we have to compute the payoff at each node, including f^u and f^d.

Put option payoff $= (K-S_T)^+$, $K = 42$



Step 3: At each node prior to maturity, decide if early exercise is optimal. Update value of contract at each of these nodes depending on if we are exercising or not.



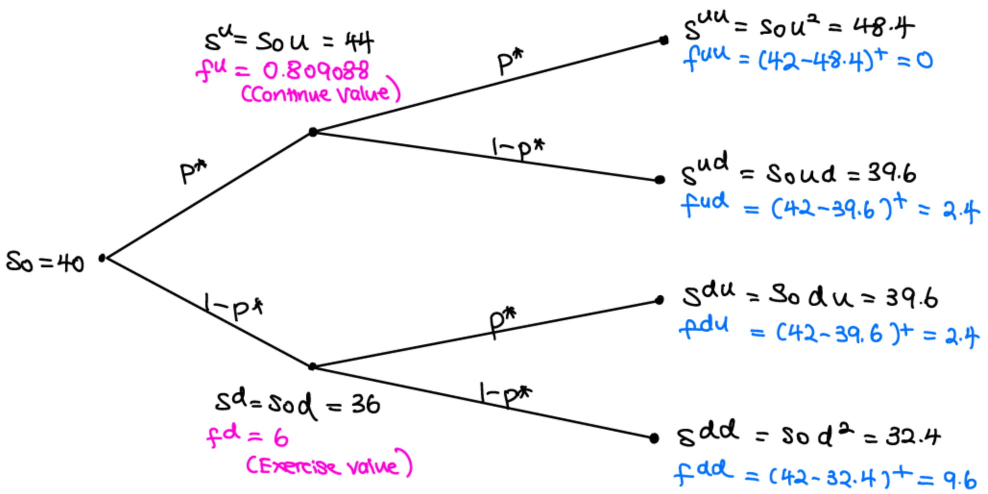
Continue Value $= e^{-r\Delta t}\left(p^*\times 0 + (1-p^*)\times 2.4\right)$
$\approx 0.80988 > 0 = \text{Payoff}^u$

Decision: Continue / Do not Exercise

Continue Value $= e^{-r\Delta t}\left(p^*\times 2.4 + (1-p^*)\times 9.6\right)$
$\approx 4.75871 < \text{Payoff } d$

Decision: Exercise!

$\downarrow$ update f^u and f^d



Remember to check at $t=0$ too!

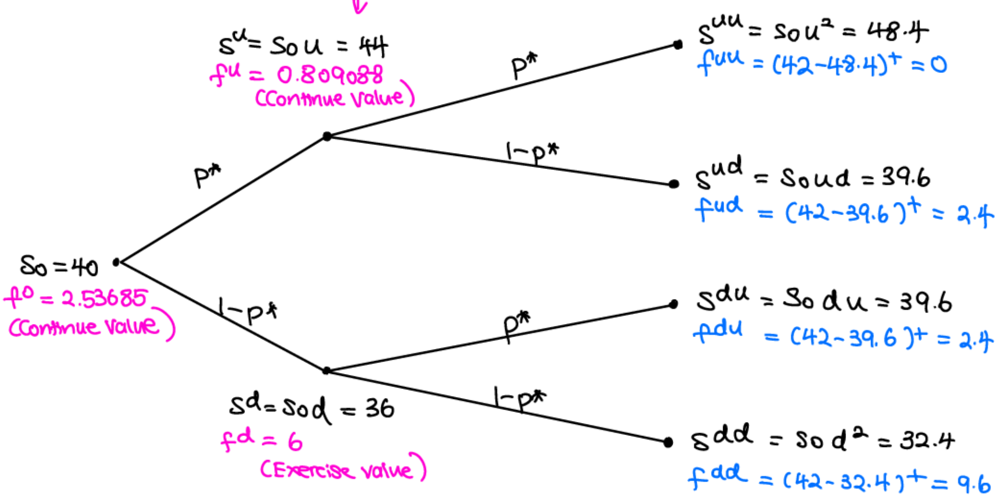
$S_0 = 40$, Payoff$^0 = (42-40)^+ = 2$.

Continue value $= e^{-r\Delta t}(p^* \cdot f^u + (1-p^*) f^d)$

$\approx 2.53685 > 2 =$ Payoff^0

$\boxed{\text{Decision: Continue / Do not Exercise}}$

$\downarrow$ update f^0



$\therefore \underbrace{f^0 \approx \boxed{\$2.54}}_{\text{Read off the value of the option at } t=0\text{.}}$

Remark: Note that early exercise in this case is optimal, which thus allows for the American put option to demand a higher price than their European counterpart.

Remark: For the European and American call options, recall that we have from Lemma 11.7 that they must have the same price (as proven in Exercise 3 of Supplement 8). Do check to see if this is true in this exercise!

Exercise 51. (Exercise 9.4.) A stock price is currently \$30. During each 2-month period for the next 4 months, it will increase by 8% or reduce by 10%. The risk-free interest rate is 5%.

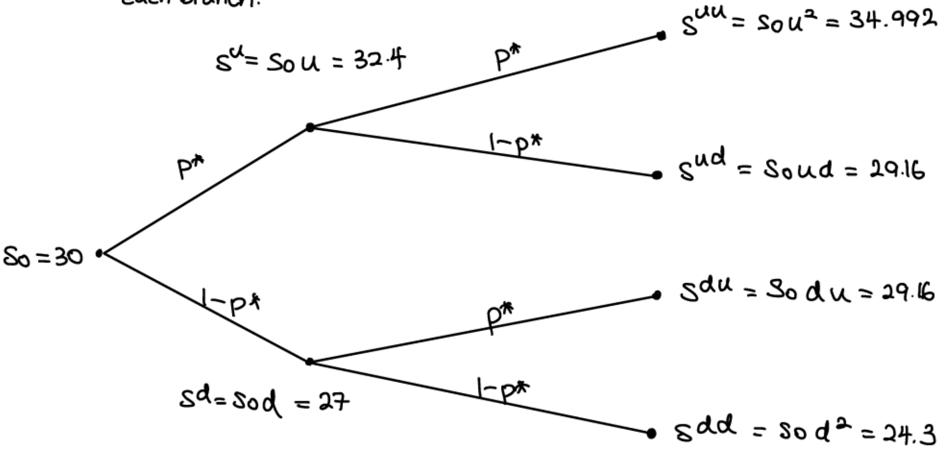
(i) Use a two-step tree to calculate the value of the derivative that pays off $(\max\{30 - S_T, 0\})^2$, where S_T is the stock price in 4 months.

(ii) If the derivative is American-style, should it be exercised early?

Suggested Solutions: This corresponds to a two-step binomial model. In this exercise, we will be using the risk-neutral valuation approach, following the steps outlined in Exercise 50. Regardless of the type of option, the risk-neutral probability can be computed (with $u = 1.08$ and $d = 0.9$) by

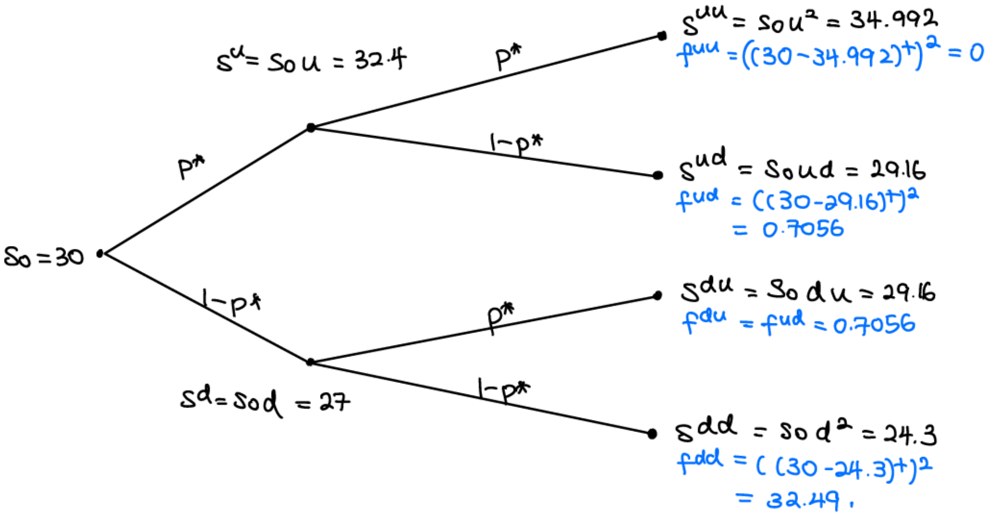
$$p^* = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05\times\frac{2}{12}} - 0.9}{1.08 - 0.9} = 0.602045.$$

Step 1. Populate the binomial tree with spot prices and the probability along each branch.



(i) Step 2 : For a European option, it can only be exercised at maturity. Hence, it suffices to compute the payoff at maturity.

Option payoff $= \max(30 - S_T, 0)^2 = ((30 - S_T)^+)^2$



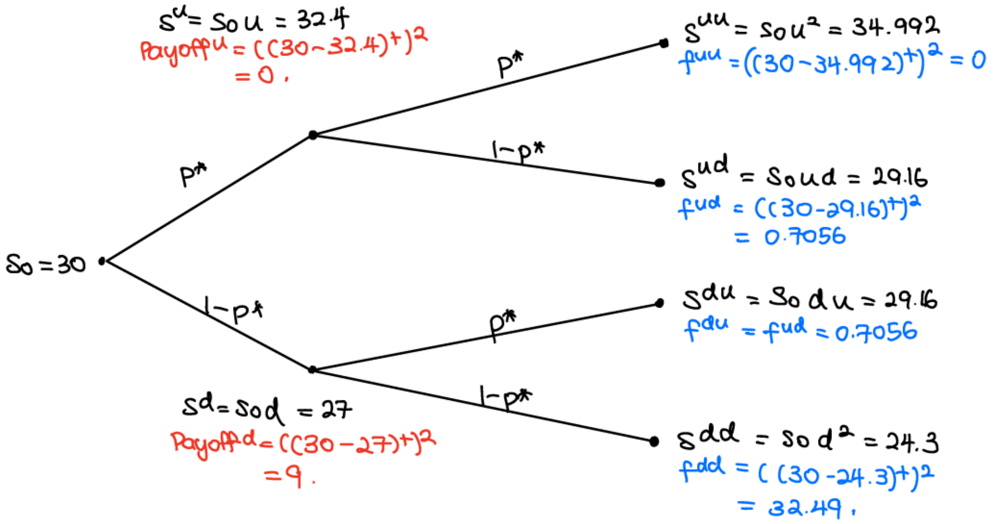
Step 3: Compute the price of the option, given by the expected discounted value of all possible payoffs. $f_0 = \mathbb{E}^*(e^{-rT}f_T)$.

$$f_0 = \mathbb{E}^*(e^{-rT}f_T) = e^{-0.05\times\frac{4}{12}} \times \Big(\overset{2p^*(1-p^*)f^{ud}}{2\times 0.602405\times(1-0.602405)\times 0.7056} + \overset{(1-p^*)^2 f^{dd}}{(1-0.602405)^2\times 32.49} \Big)$$

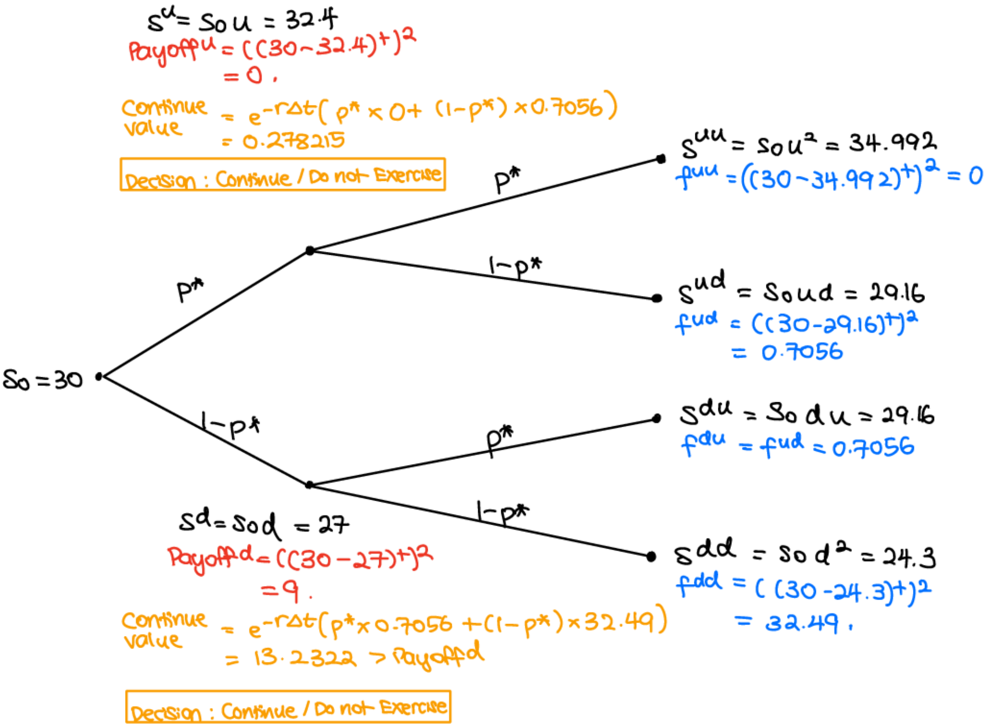
$$\approx \boxed{\$5.38}$$

(ii) Step 2: For an American option, it can be exercised at anytime up to maturity. Hence, we have to compute the payoff at each node, including f^u and f^d.

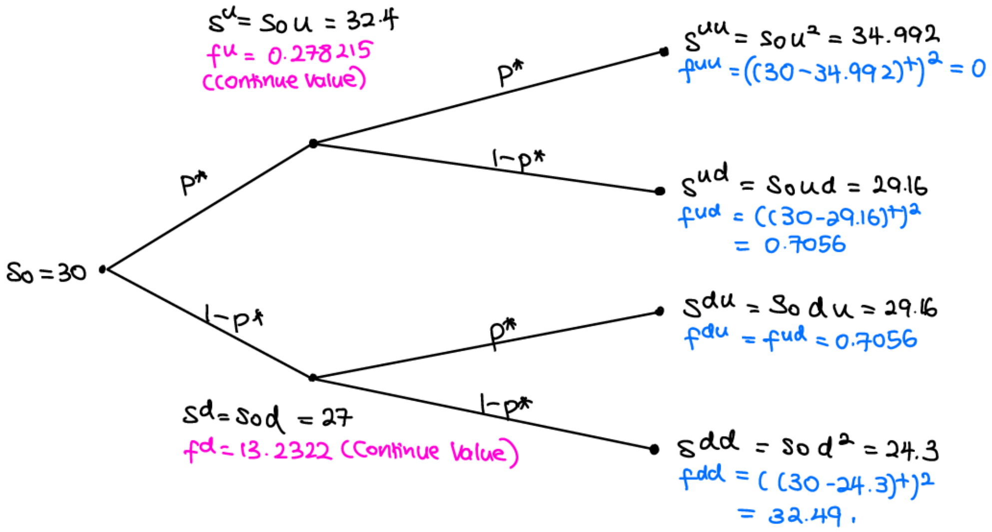
Option payoff $= \max(30-S_T, 0)^2 = ((30-S_T)^+)^2$



Step 3: At each node prior to maturity, decide if early exercise is optimal. Update value of contract at each of these nodes depending on if we are exercising or not.



$\downarrow$ update f^u and f^d



At $t=0$, $\left.\begin{array}{l}\text{payoff}^0 = ((30-30)^+)^2 = 0 \\ \text{continue value} > 0 .\end{array}\right\}$ $\therefore$ Do not exercise.

$f^0 = $ continue value

$= e^{-r\Delta t}(p^* \times 0.278215 + (1-p^*) \times 13.2322)$

$\approx \boxed{\$5.38}$

Remark: In this case, since it is not optimal to exercise the American-style option before maturity, it should then be priced the same as the European-style counterpart.

10 Discussion 10

Brownian Motion, Wiener Process, and Black-Scholes-Merten Model.

Before we begin, let us recap a couple of properties of continuous random variables (and in particular, normal distributions). Let $X \sim N(\mu, \sigma^2)$, where μ and σ^2 represent the mean and variance of the normal distribution. We then have the following properties:

- Mean and Variance: $\mathbb{E}(X) = \mu, \mathrm{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \sigma^2$.
- $\mathbb{E}(X^2) = \mathbb{E}(X)^2 + \mathrm{Var}(X)$.
- $\mathbb{P}(X = a) = 0$ for $a \in \mathbb{R}$ (the probability of a continuous random variable at exactly a value is zero).
- If X and Y are independent random variables, then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.
- For any two random variables, we have $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.
- Probability density function (pdf):
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}.$$
- Cumulative distribution function (cdf) at $x \in \mathbb{R}$:
$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(z)\mathrm{d}z.$$
- Computing probabilities:
$$\mathbb{P}(a \leq X \leq b) = \int_{a}^{b} f(z)\mathrm{d}z = F(b) - F(a).$$
- Computing expected values:
$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)\mathrm{d}x.$$
- Addition and Scaling. If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$, then for any $a, b \in \mathbb{R}$,
$$aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2).$$
- Standardization. Let $Z \sim N(0, 1)$. Then, for a given $X \sim N(\mu, \sigma^2)$, we can standardize it into Z as follows:
$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1).$$
- Moment Generating Function (MGF) for $u \in \mathbb{R}$ and $X \sim N(\mu, \sigma^2)$:
$$\mathbb{E}(e^{uX}) = e^{\mu u + \frac{1}{2}\sigma^2 u^2}.$$
In particular, we can use this to compute $\mathbb{E}(e^X)$ by setting $u = 1$ to obtain
$$\mathbb{E}(e^X) = e^{\mu + \frac{1}{2}\sigma^2}.$$

A **standard Brownian motion** on $\mathbb{R}$ is a continuous-time stochastic process $(B_t)_{t \geq 0}$ satisfying the following properties:

- $B_0 = 0$.
- (Independent increments.) For any $0 \leq t_1 < t_2 < t_3$, the random variables $B_{t_2} - B_{t_1}$ and $B_{t_3} - B_{t_2}$ are independent.
- (Stationary increments.) For any $0 \leq s < t$, we have
$$B_t - B_s \sim N(0, t - s).$$
- The function $t \mapsto B_t$ is continuous.

In particular, we have

- $B_t \sim N(0, t)$ for $t > 0$.

Next, let $(B_t)_{t\geq 0}$ be a standard Brownian motion. The process $(X_t)_{t\geq 0}$ defined by

$$X_t = x_0 + \mu t + \sigma B_t (t \geq 0)$$

is called the **Brownian motion with drift parameter** μ and **variance parameter** σ^2 (with $x_0, \mu \in \mathbb{R}, \sigma > 0$). This is also known as the **generalized Wiener process**. In particular,

- $X_t \sim N(x_0 + \mu t, \sigma^2 t)$ for $t \geq 0$.
- Stationary and independent increments, with

  $$X_t - X_s \overset{d}{=} X_{t-s} - X_0 \sim N(\mu(t-s), \sigma^2(t-s))$$

  and $X_t - X_s$ is independent of X_s for $0 < s < t$.
- The standard Brownian motion corresponds to $x_0 = 0, \mu = 0$, and $\sigma^2 = 1$.

Note that this is used to model stock prices. For instance, under the **Black-Scholes-Merton** model, let $(B_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ represent the standard Brownian motion and the stock price process $(S_t)_{t\geq 0}$. The price of the stock at time $t \geq 0$ is modeled by

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)\cdot t + \sigma B_t}.$$

Here,

- $S_0 =$ Spot price at $t = 0$.
- $\mu =$ **Expected rate of return** of a stock (per annum).
- $\sigma (> 0) =$ Stock price's **volatility** (per annum).

Note that a stochastic process modeled this way is also known as a **geometric Brownian motion**.

Exercise 52. (Exercise 10.1.) Let $(B_t)_{t\geq 0}$ be a standard Brownian motion.

(i) For $t > 0$, what is the distribution of B_t?

(ii) For $0 < s < t$, find the distribution of $B_t + B_s$.

(iii) For $s, t > 0$, find the covariance of B_s and B_t.
Recall that $\text{Cov}(X, Y) = \mathbb{E}\left((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\right).$

Suggested Solutions:

(i) $B_t \sim N(0, t)$.

(ii) Observe that

$$\begin{aligned} B_s + B_t &= B_t - B_s + 2B_s \\ &\sim N(0, t-s) + 2N(0, s) \quad \text{(independent)} \\ &\sim N(0, (t-s) + 2^2 s) \\ &\sim N(0, t + 3s). \end{aligned}$$

Here, we have used the fact that $B_t - B_s$ is independent of B_s by the independent increment property of Brownian motion.

(iii) Since $\mathbb{E}(B_s) = \mathbb{E}(B_t) = 0$, we have $\text{Cov}(B_t, B_s) = \mathbb{E}(B_t B_s)$. Now, assuming that $s < t$, since $B_t = B_s + (B_t - B_s)$, we have

$$\begin{aligned} \mathbb{E}(B_t B_s) &= \mathbb{E}((B_s + (B_t - B_s))B_s) \\ &= \mathbb{E}(B_s^2 + B_s(B_t - B_s)) \\ &= \mathbb{E}(B_s^2) + \mathbb{E}(B_s(B_t - B_s)). \end{aligned}$$

Next, observe that by the independent increment property of Brownian motion, B_s and $B_t - B_s$ are independent (for $t > s$), hence, we have

$$\mathbb{E}(B_s(B_t - B_s)) = \mathbb{E}(B_s) \cdot \mathbb{E}(B_t - B_s) = 0 \cdot 0 = 0.$$

On the other hand, we have

$$\mathbb{E}(B_s^2) = \mathbb{E}(B_s)^2 + \text{Var}(B_s) = 0^2 + s = s.$$

Henceforth, we have

$$\text{Cov}(B_t, B_s) = \mathbb{E}(B_t B_s) = s + 0 = s$$

if $t > s$. Using a similar argument, if $t < s$, we have

$$\text{Cov}(B_t, B_s) = t.$$

Hence, for any $t, s > 0$, we have

$$\text{Cov}(B_t, B_s) = \min\{s, t\}.$$

Exercise 53. (Exercise 10.2.) A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.5 per quarter and a variance rate of 4.0 per quarter. How high does the company's initial cash position have to be to have a 5% chance of a negative cash position in one year?

Hint: Use the fact that $\mathbb{P}(Z \leq -1.645) = 0.05$ for $Z \sim N(0,1)$.

Suggested Solution: Let t be the number of quarters elapsed, and X_t to be a company's cash position. Then, with $\mu = 0.5$ and $\sigma^2 = 4.0$, we have

$$X_t \sim N(x_0 + \mu t, \sigma^2 t) = N(x_0 + 0.5t, 4t).$$

For the company's cash to be negative in one year with 5% chance, this is equivalent to

$$\mathbb{P}(X_4 \leq 0) = 0.05.$$

Observe that

$$X_4 \sim N(x_0 + 2, 16).$$

To utilize the hint, we **standardize** the normal distribution as follows:

$$\begin{aligned} \mathbb{P}(X_4 \leq 0) &= 0.05, \quad X_4 \sim N(x_0 + 2, 16). \\ \mathbb{P}(X_4 - (x_0 + 2) \leq -(x_0 + 2)) &= 0.05, \quad X_4 - (x_0 + 2) \sim N(0, 16). \\ \mathbb{P}\left(\frac{X_4 - (x_0 + 2)}{4} \leq \frac{-(x_0 + 2)}{4}\right) &= 0.05, \quad \frac{X_4 - (x_0 + 2)}{4} \sim N\left(0, \frac{1}{4^2}16\right) = N(0,1) \sim Z. \\ \mathbb{P}\left(Z \leq \frac{-(x_0 + 2)}{4}\right) &= 0.05, \quad Z \sim N(0,1). \end{aligned}$$

By the hint, we have

$$-\frac{x_0 + 2}{4} = -1.645.$$

Solving for x_0, we have

$$\boxed{x_0 = 4.58}.$$

Hence, the company's initial cash position should be at 4.58 million dollars.

Exercise 54. (Exercise 10.3.) In the famous Black-Sholes-Merton model (1973), the price of the stock $(S_t)_{t\geq 0}$ at time $t \geq 0$ with initial spot price S_0 (today at time $t = 0$) is modeled by

$$S_t = S_0 \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right)\cdot t + \sigma \cdot B_t}, \tag{9}$$

where $B_t \sim N(0, t)$ corresponds to the underlying Brownian motion driving the stock price process. The parameter $\mu \in \mathbb{R}$ represents the expected return of the stock (per annum) and $\sigma > 0$ is the stock price's volatility (per annum).

(i) Compute the expected value $\mathbb{E}(S_t)$ of the stock price S_t at time t given in (9).
Hint: Use the moment generating function for a normal distribution.

(ii) Let $r > 0$ represent the risk-free rate (p.a. and continuously compounded). How does your result from (i) change if the parameter μ is replaced by r in (9)? What is the interpretation of this change?

Suggested Solutions:

(i) To do so, we first let $Y_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t$ and compute the parameters of this normal distribution. Since $B_t \sim N(0, t)$, we have $\sigma B_t \sim N(0, \sigma^2 t)$ and hence

$$Y_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right).$$

Furthermore, recall that if $W \sim N(\tilde{\mu}, \tilde{\sigma}^2)$, then $\mathbb{E}(e^{uW}) = e^{\tilde{\mu}u + \frac{1}{2}\tilde{\sigma}^2 u^2}$. For $u = 1$, we have

$$\mathbb{E}(e^W) = e^{\tilde{\mu} + \frac{\tilde{\sigma}^2}{2}}.$$

Hence, we have

$$\begin{aligned} \mathbb{E}(S_t) &= \mathbb{E}(S_0 e^{Y_t}) \\ &= S_0 \mathbb{E}(e^{Y_t}) \\ &= S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \frac{\sigma^2 t}{2}} \\ &= S_0 e^{\mu t}. \end{aligned}$$

(ii) If we set $\mu = r$, then

$$\mathbb{E}(S_t) = S_0 e^{rt}.$$

This implies that for a risk-free asset with the expected rate of return to be at the risk-free interest rate, the expected stock price at time $t > 0$ will be equivalent to investing S_0 (= spot price of the stock at $t = 0$) amount of cash in a bank at the risk-free interest of r.

Exercise 55. (Exercise 10.4.) A stock price is currently \$50. Its expected return and volatility are 12 % and 30% p.a. respectively. What is the probability (under the Black-Scholes-Merton model assumption (9)) that the stock price will be greater than \$80 in 2 years?

Hint: Express this probability in terms of the cumulative distribution function $\Phi(x)$ of the standard normal distribution, that is, $\Phi(x) = \mathbb{P}(Z \leq x)$ for $Z \sim N(0,1)$.

Suggested Solutions:

By the Black-Scholes-Merton model, we have

$$S_t = S_0 e^{Y_t}$$

with

$$Y_t = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right).$$

Since $\mu = 0.12$ and $\sigma = 0.3$, we have

$$Y_t \sim N(0.075t, 0.09t).$$

For 2 years, we have $t = 2$ and thus

$$Y_2 \sim N(0.15, 0.18).$$

With $S_0 = 50$, the required probability is given by

$$\begin{aligned}
\mathbb{P}(Y_2 \geq 80) &= \mathbb{P}(50e^{Y_2} \geq 80) \\
&= \mathbb{P}(e^{Y_2} \geq 1.6) \\
&= \mathbb{P}(Y_2 \geq \ln(1.6)) \\
&= \mathbb{P}\left(Y_2 - 0.15 \geq \ln(1.6) - 0.15\right); \quad Y_2 - 0.15 \sim N(0, 0.18) \\
&= \mathbb{P}\left(\frac{Y_2 - 0.15}{\sqrt{0.18}} \geq \frac{\ln(1.6) - 0.15}{\sqrt{0.18}}\right); \quad \frac{Y_2 - 0.15}{\sqrt{0.18}} \sim N\left(0, \frac{1}{(\sqrt{0.18})^2}0.18\right) = N(0,1) \sim Z \\
&= \mathbb{P}\left(Z \geq \frac{\ln(1.6) - 0.15}{\sqrt{0.18}}\right) \\
&= 1 - \mathbb{P}\left(Z \leq \frac{\ln(1.6) - 0.15}{\sqrt{0.18}}\right) \\
&= 1 - \Phi\left(\frac{\ln(1.6) - 0.15}{\sqrt{0.18}}\right) \\
&\approx 0.2253.
\end{aligned}$$

Note that $\Phi(\cdot)$ is to be evaluated numerically on a calculator or by checking the cdf table for a standard normal distribution.

Exercise 56. (Exercise 10.5.) Suppose that a stock price has an expected return of 16% p.a. and volatility of 30% p.a. When the stock price at the end of a certain day is \$50, calculate (under the Black-Scholes-Merton model assumption (9)) the following (note that time is measured in years; hence, one trading day corresponds to 1/252 years because there are a total of 252 days per year when the stock exchange is open):

(i) The expected stock price at the end of the next day.
Hint: Use Exercise 10.3(i).

(ii) The standard deviation of the stock at the end of the next day.
Hint: Use Exercise 10.3(i) and $\mathrm{Var}(S_t) = \mathbb{E}(S_t^2) - (\mathbb{E}(S_t))^2$.

(iii) The 95% confidence interval for the stock price at the end of the next day.
Hint: Use the fact that

$$\mathbb{P}(-1.96 < Z < 1.96) = 0.95$$

for $Z \sim N(0,1)$.

Suggested Solutions:

(i) $\mathbb{E}(S_t) = S_0 e^{\mu t} = 50e^{0.16\times\frac{1}{252}} \approx \$50.0318$.

(ii) Recall that $S_t = S_0 e^{Y_t}$ with $Y_t \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$. To use the hint, one would have to know how to compute $\mathbb{E}(S_t^2)$. To so do, observe that

$$S_t^2 = S_0^2 e^{2Y_t} = S_0^2 e^{V_t}$$

with

$$V_t = 2Y_t \sim N\left(2\left(\mu - \frac{\sigma^2}{2}\right)t, 4\sigma^2 t\right).$$

Hence, using a similar strategy as in Exercise 10.3(i), since $W \sim N(\tilde{\mu}, \tilde{\sigma}^2)$ implies

$$\mathbb{E}(e^W) = e^{\tilde{\mu}+\frac{\tilde{\sigma}^2}{2}},$$

then

$$\mathbb{E}(e^{V_t}) = e^{2\left(\mu-\frac{\sigma^2}{2}\right)t+\frac{4\sigma^2 t}{2}} = e^{2\mu t+\sigma^2 t}.$$

This then implies that

$$\begin{aligned}\mathrm{Var}(S_t) &= \mathbb{E}(S_t^2) - \mathbb{E}(S_t)^2\\ &= S_0^2\mathbb{E}(e^{V_t}) - \mathbb{E}(S_t)^2\\ &= S_0^2 e^{2\mu t+\sigma^2 t} - (i)^2\\ &= 50^2 e^{2\times 0.16\times\frac{1}{252}+0.3^2\times\frac{1}{252}} - (50.0318)^2\\ &\approx 0.88976.\end{aligned}$$

Hence, the standard deviation of stock price at the end of the next day is given by $\sqrt{\mathrm{Var}(S_t)} \approx \$0.9433$.

(iii) Recall that $S_t = S_0 e^{Y_t}$, with $t = \frac{1}{252}$ years, $\mu = 0.16$ and $\sigma = 0.3$, we have

$$Y_t \sim N\left(\left(0.16 - \frac{0.3^2}{2}\right)\cdot\frac{1}{252}, 0.3^2 \times \frac{1}{252}\right) = N\left(0.000456349, 0.000357143\right).$$

Hence,

$$\frac{Y_t - 0.000456349}{\sqrt{0.000357143}} = Z \sim N(0,1).$$

Observe that by “reversing the standardization process”, we have

$$\begin{aligned}
\mathbb{P}(-1.96 < Z < 1.96) &= 0.95 \\
\mathbb{P}\left(-1.96 < \frac{Y_t - 0.000456349}{\sqrt{0.000357143}} < 1.96\right) &= 0.95 \\
\mathbb{P}\left(-0.0370405 < Y_t - 0.000456349 < 0.0370405\right) &= 0.95 \\
\mathbb{P}\left(-0.0365842 < Y_t < 0.0374969\right) &= 0.95 \\
\mathbb{P}\left(e^{-0.0365842} < e^{Y_t} < e^{0.0374969}\right) &= 0.95 \\
\mathbb{P}\left(50e^{-0.0365842} < S_t = S_0 e^{Y_t} < 50e^{0.0374969}\right) &= 0.95. \\
\mathbb{P}\left(48.2038 < S_t < 51.8868\right) &= 0.95.
\end{aligned}$$

Hence, the 95% confidence interval for the stock price at the end of the next day is given by

$$\boxed{S_t \in (48.2038, 51.8868)}.$$

References

[1] John C. Hull. *Options, Futures, and Other Derivatives*. Pearson, 11th edition, 2018.