**Math 266 - Discussion Supplements for Winter 24**

Contents are motivated from [1], [2], and [3]. [1]

It is worth noting that most of the problems discussed are problems from past qualifying exams for the Applied Differential Equations qualifying exam. You can access them here. At the end of each discussion, I have included relevant qual problems which you can practice if you're studying for the quals (which should be either doable or "within reach" with what we have covered, though I can't guarantee that it is the case since I have not solved every single past qual problems). If I'm still around in UCLA and you have questions with regards to any of them, drop me an email and we'll talk!

For reference, the following is a legend for a bunch of ducks that you are going to see:

signifies that we are moving on to (optional) remarks relevant to what we have just discussed.

signifies that you should be cautious of what precedes the duck!

refers to optional materials that are related to the qualifying exam but might not be tested directly in the midterm/final exam for this class.

] refers to optional materials that are **not** (or highly likely not) tested in the qualifying exam but are concepts relevant to the homework problems for this class.

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# 1 Discussion 1

Review: First-order linear PDEs and Method of Characteristics.
(References: Shearer and Levy Chapters 1 and 3.)

Given a first order linear PDE, for an unknown function $u(x, y)$ of two variables, of the form

$$a(x, y)u_x + b(x, y)u_y = 0.$$

To solve the equation above, we employ the method of characteristics as follows. The main idea here is that along a parameterized curve, the above PDE then reduces to an ODE (or possibly a system of ODEs), which can then be solved easily.

This idea is better illustrated with an example, which we shall see below.

**Example 1.** Solve the following first order PDE using method of characteristics

$$yu_x + u_y = 0, \quad u(x,0) = x^3 \text{ for all } x \in \mathbb{R}, \quad \text{for } x, y \in \mathbb{R}. \tag{1}$$

Suggested Solutions:

<u>Method 1:</u> Considering a parametrized curve of the form $(x, y(x))$.

First, note that if $y = 0$, then $u(x,0)$ is just as given above. Thus, we will consider a given $(x,y) \in \mathbb{R}^2$ such that $y \neq 0$. From the PDE, this implies that

$$u_x + \frac{1}{y} u_y = 0. \tag{2}$$

Now, by writing $u(x,y) = u(x, y(x))$, the total derivative with respect to $x$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}x} u(x, y(x)) = \frac{\partial u}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\partial u}{\partial y} \underbrace{=}_{\text{by PDE (2)}} 0 \tag{3}$$

if we set

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y}. \tag{4}$$

Solving (4) yields

$$\frac{y^2}{2} = x + C \implies y^2(x) - 2x = C' \tag{5}$$

for some constants[2] $C$ and $C'$. Hence, observe that

1. The characteristic curve is given in (5) as the solution to the ODE (4).
2. (3) tells us that $\frac{\mathrm{d}}{\mathrm{d}x} u(x, y(x)) = 0$, which implies that along the characteristic curves parametrized by $(x, y(x))$, $u$ is constant.

Now, given any $(x,y) \in \mathbb{R}^2$ (with $y \neq 0$), this must lie on some characteristic curve. To determine the characteristic curve that $(x,y)$ lies on, it suffices to determine the value of $C'$ in (5). This implies that

$$C' = y^2 - 2x \tag{6}$$

is the corresponding parameter that gives the characteristic curve that passes through $(x,y)$. Tracing along this characteristic curve back to $y = 0$, by (5), this happens when

$$0 - 2x = C' \implies x = -\frac{C'}{2}.$$

Now since $u$ is constant along the characteristic curve $(x, y(x))$, we have

$$\boxed{u(x,y) \underbrace{=}_{\text{along char}} u(x, y(x)) \underbrace{=}_{u \text{ constant along char}} u\left(-\frac{C'}{2}, 0\right) \underbrace{=}_{\text{initial data}} \left(-\frac{C'}{2}\right)^3 \underbrace{=}_{(6)} -\frac{1}{8}(y^2 - 2x)^3}.$$

[2] Here, we absorb 2 into $C$ and write it as $C'$ for convenience.

Method 2: Considering a parameterized curve of the form $(x(s), y(s))$ (with an additional parameter $s$). Recall that the PDE is given by

$$yu_x + 1u_y = 0 \tag{7}$$

By writing $u(x, y) = u(x(s), y(s))$, the total derivative with respect to $s$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}s}u(x(s), y(s)) = \frac{\mathrm{d}x(s)}{\mathrm{d}s}\frac{\partial u}{\partial x}(x(s), y(s)) + \frac{\mathrm{d}y(s)}{\mathrm{d}x}\frac{\partial u}{\partial y}(x(s), y(s)) \underbrace{=}_{\text{by PDE (7)}} 0 \tag{8}$$

if we set

$$\begin{cases} \frac{\mathrm{d}x(s)}{\mathrm{d}s} = y(s), \\ \frac{\mathrm{d}y(s)}{\mathrm{d}s} = 1. \end{cases} \tag{9}$$

Solving the system of ODEs in (9), we have

$$\begin{cases} x(s) = x(0) + y(0)s + \frac{s^2}{2}, \\ y(s) = y(0) + s. \end{cases} \tag{10}$$

Now, given any $(x, y) \in \mathbb{R}^2$ (with $y \neq 0$), suppose that we would want to connect to $(x, y)$ via the characteristic curve at parameter value $s$ from the initial data along $y = 0$ (with $s = 0$ corresponds to starting along the initial data). This implies that $y(0) = 0$ and $x(0) = x_0$ for some $x_0 \in \mathbb{R}$ (to be determined). In other words:

- $(x(s), y(s)) = (x, y)$, and
- $(x(0), y(0)) = (x_0, 0)$.

Plugging these into our solution in (10), the second equation implies $y = s$. Substituting this into the first equation, we then obtain

$$x = x_0 + \frac{y^2}{2} \implies x_0 = x - \frac{y^2}{2}. \tag{11}$$

Furthermore, recall that (8) implies that $u$ is constant along characteristics. Starting from $(x, y)$ and tracing along the characteristic curve with $x_0$ determined in (11), similar to Method 1, we have

$$\boxed{u(x, y) \underbrace{=}_{\text{along char}} u(x(s), y(s)) \underbrace{=}_{u \text{ constant along char}} u(x_0, 0) \underbrace{=}_{(11)} u\left(x - \frac{y^2}{2}, 0\right) \underbrace{=}_{\text{initial data}} \left(x - \frac{y^2}{2}\right)^3 = -\frac{1}{8}(y^2 - 2x)^3}.$$

□

Remark: In physical terms, by an appropriate change of frame of reference (ie coordinates), we can "reduce" the first-order PDE to a system of ODE. If $u$ represents the amount of some quantity, $x$ represents time, and $y$ represents (one-dimensional) space, then we can think of the characteristic curves as "following the flow of a quantity", in which the evolution on the quantity will be governed by an ODE (with additional ODE describing how the coordinates should change to "keep up" with the flow). On the other hand, we can think of the PDE as the description of the evolution/flow of a quantity $u$ in a stationary frame of reference (coordinate system). In fluid dynamics, these correspond to:

- **Eulerian coordinates:** Recording the flow of a quantity in a fixed frame of referece.
- **Lagrangian coordinates:** Recording the flow of a quantity with a frame of reference that is moving along with the flow of the quantity itself.

This is illustrated better with the derivation for the solution to a first order linear PDE with constant coefficients below:

**Example 2.** Solve the following first order PDE using method of characteristics

$$u_t + au_x = 0, \quad u(0,x) = u_0(x) \text{ for all } x \in \mathbb{R}, \qquad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}.$$

Suggested Solutions: Considering a parametrized curve of the form $(t, x(t))$. The PDE is given by

$$u_t + au_x = 0.$$

Now, by writing $u(t,x) = u(t,x(t))$, the total derivative with respect to $t$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t,x(t)) = \frac{\partial u}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial u}{\partial x} \underbrace{=}_{\text{PDE}} 0$$

if we set

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a.$$

Solving the ODE above yields

$$x = at + C. \tag{12}$$

for some constants $C$, with $u$ constants along characteristics. Given any $(t,x) \in (0,\infty) \times \mathbb{R}$, from (12), we have that the characteristic curve that passes through $(t,x)$ has the following value of $C$:

$$C = x - at.$$

Tracing along this characteristic curve back to $t = 0$, by (12), this happens when

$$x = C.$$

Now since $u$ is constant along the characteristic curve $(t, x(t))$, we have

$$\boxed{u(t,x) \underbrace{=}_{\text{along char}} u(t,x(t)) \underbrace{=}_{u \text{ constant along char}} u(0,C) \underbrace{=}_{\text{initial data}} u_0(C) = u_0(x - at).}$$

□

Remark: In the example above, the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a$$

represents the speed of the "moving reference frame" (measured with respect to the canonical/stationary $(t,x)$ coordinates).

## Additional Considerations for Method of Characteristics.

We might face potential problems when we are trying to solve a PDE by the method of characteristics. These includes

- Multiple values at a given point. This can happen if there exists **two (possibly) different** characteristics with different values of (constant) $u$ along these characteristics connecting to the given point.
  (In mathematical terms, this corresponds to a **lack of uniqueness** of solutions to the PDE.)
- No possible value at a given point. This happens if there are **no** characteristics from the auxiliary boundary conditions that connects to the given point.
  (In mathematical terms, this corresponds to a **lack of existence** of solutions to the PDE.)

Let us explore this and several other related phenomena in the example below.

**Example 3.** Consider the PDE consisting of the unknown function $u(x, y)$ given by

$$u_x + 3x^2 u_y = u,$$

with boundary conditions

$$\begin{cases} u(x,0) = x^3 & \text{for all } x \geq 0 \\ u(0,y) = y^2 & \text{for all } y \geq 0. \end{cases}$$

(i) Solve for $u(x, y)$ for $x, y > 0$.

(ii) What happens if we do not impose the boundary conditions along $y = 0$?

(iii) Note that your solution in (i) is in general not $C^1$ across the curve $y = x^3$. What can you attribute this phenomenon to?
(Recall that $C^1$ here means once continuously differentiable.)

Suggested Solutions:

(i) Consider the parametrized characteristic curve given by $(x(s), y(s))$. Writing down the total derivative, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} u(s) \equiv \frac{\mathrm{d}}{\mathrm{d}s} u(x(s), y(s)) = \frac{\mathrm{d}x(s)}{\mathrm{d}s}\frac{\partial u}{\partial x} + \frac{\mathrm{d}y(s)}{\mathrm{d}s}\frac{\partial u}{\partial y} = u(s) \tag{13}$$

by the PDE if we set

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}s} = 1, \\ \frac{\mathrm{d}y}{\mathrm{d}s} = 3x^2. \end{cases}$$

Coupled with the ODE for $u(s)$ in (13), we have

$$\begin{cases} \frac{\mathrm{d}x(s)}{\mathrm{d}s} = 1, \\ \frac{\mathrm{d}y(s)}{\mathrm{d}s} = 3x^2(s), \\ \frac{\mathrm{d}u(s)}{\mathrm{d}s} = u(s). \end{cases} \tag{14}$$

The solution to the first equation is given by

$$x(s) = x(0) + s, \tag{15}$$

while the solution to the third equation is given by

$$u(s) = u(0)e^s. \tag{16}$$

Plugging $x(s)$ into the second equation and solving the ODE for $y(s)$, we have

$$y(s) = y(0) + (x(0) + s)^3 - x(0)^3. \tag{17}$$

Note that there are **two** boundary conditions given. We shall start with the boundary condition along $x = 0$ ($y-$axis). For this boundary condition, for any given $(x, y)$ for $x, y > 0$, we would like to trace along a characteristic curve $(x(s), y(s))$ such that

- $(x(0), y(0)) = (0, y_0)$ (for some $y_0 > 0$ to be determined), and
- $(x(s), y(s)) = (x, y)$.

Plugging these conditions into (15) and (17), we can then determine that

$$y_0 = y - x^3, \quad s = x. \tag{18}$$

Together with the evolution of $u$ along characteristics (16), this then implies that

$$u(x, y) = u(x(s), y(s)) \underbrace{=}_{(16)} u(0, y_0)e^s = u\left(0, y - x^3\right) e^x = (y - x^3)^2 e^x. \tag{19}$$

Note that (19) only works if $y_0 = y - x^3 > 0$. This implies that for points with $y - x^3 \leq 0$, we might have to utilize the boundary conditions along $y = 0$ ($x-$axis).

To do so, starting from $(0, x_0)$ up to $(x(s), y(s)) = (x, y)$, we have from (15) and (17),

$$x_0 = (x^3 - y)^{1/3}, \quad s = x - x_0 = x - (x^3 - y)^{1/3}. \tag{20}$$

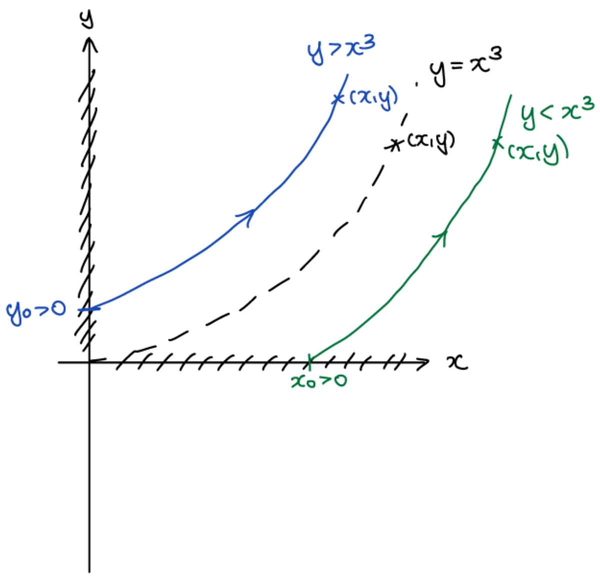
Here, $x_0$ is now positive if $y - x^3 < 0$.
This then implies

$$u(x,y) = u(x(s), y(s)) = u(x(0), y(0))e^s = u\left((x^3 - y)^{1/3}, 0\right) e^{x-(x^3-y)^{\frac{1}{3}}} = -(y - x^3)e^{x-(x^3-y)^{\frac{1}{3}}}. \tag{21}$$

This time, we see that $x_0 > 0$ if $y_0 < 0$. This implies that applying the boundary condition for $u(\cdot, 0)$ along the positive $x-$axis is valid. Henceforth, we obtain the solution for the entire domain $x, y > 0$ as follows:[3]

$$\boxed{u(x,y) = \begin{cases} (y - x^3)^2 e^x & \text{for} \quad y \geq x^3, \\ -(y - x^3)e^{x-(x^3-y)^{\frac{1}{3}}} & \text{for} \quad y \leq x^3. \end{cases}} \tag{22}$$

The following diagram summarizes the above computations:




(ii) As seen in the diagram and computations above, not imposing boundary conditions along $y = 0$ would imply that solutions in the region $y < x^3$ might not exists as no characteristics pass through points in that region.

(iii) This can be attributed to the **lack of regularity of the boundary**. In the equation above, the curve in which we prescribe boundary conditions along has a "kink" at the origin.
(In mathematical terms, we say that the boundary is not necessarily smooth, though its possibly Lipschitz.)
Note that this **cannot be attributed to lack of smoothness across the initial data**, since the initial data is actually $C^1$ along the two different paths (along the axes) to the origin.

□

[3] Note that along $y = x^3$, we must have $y_0 = 0$ and $x_0 = 0$. Hence, $u(x_0, y_0) = 0$, which from (16) implies that $u(x,y) = 0$ for all points along the curve $y = x^3$. This is consistent with (22), in which $u(x,y)$ from both sides of the curve $y = x^3$ vanishes.

Qual problem(s) for additional practice:

**Exercise 1.** (Spring 22, Problem 6, Hyperbolic.)
Using the method of characteristics, solve

$$xu_x + yu_y = u, \quad u(x,1) = \frac{1}{1+x^2}$$

for $u(x,y)$ with $y > 0$.

## 2 Discussion 2

<u>Semilinear first-order PDEs.</u>
(References: Shearer and Levy Chapters 3.3, 3.4)

In this discussion, we will look at two different types of nonlinear first-order PDEs. The first is known as the semilinear first-order PDEs, with the following general form:

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u).$$

The method of characteristics with the appropriate boundary/initial conditions covered in Discussion 1 can be employed directly to solve the PDE on its domain of interest. Henceforth, similar concerns with regards to the lack of existence and uniqueness of solutions to this PDE carry over too. In particular, recall that we might face the following issues:

- Multiple values at a given point. This can happen if there exists <u>**two (possibly) different**</u> characteristics with different values of (constant) $u$ along these characteristics connecting to the given point.
  (In mathematical terms, this corresponds to a **lack of uniqueness** of solutions to the PDE.)
- No possible value at a given point. This happens if there are <u>**no**</u> characteristics from the auxiliary boundary conditions that connects to the given point.
  (In mathematical terms, this corresponds to a **lack of existence** of solutions to the PDE.)

In fact, when two characteristics collide, we say that the solutions *break down* at that point. Without loss of generality, we consider an equation of the form:

$$u_t + b(t,x,u)u_x = c(t,x,u). \tag{23}$$

To determine this point in which the solutions break down, we can either:

- Obtain the expressions for the characteristic curves directly and see if these curves actually collide; or
- Following Shearer and Levy's Book Section 3.4, since solutions break down at the time in which two characteristics carrying different values of $u$ collide, we must have $u_x \to \infty$. To investigate when this happens, we take the partial derivative with respect to $x$ on (23) and investigate the resulting PDE for $p := u_x$.

We will see more examples of investigating solutions that break down via the first method in a future discussion. For now, we will illustrate the second method in an example below.

**Example 4.** Consider the forced invisid Burgers' equation for an unknown quantity $u(t,x)$:

$$\begin{cases} u_t + uu_x = 1 - u & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = g(x) & \text{for } (t,x) \text{ on } \{t=0\} \times \mathbb{R}. \end{cases}$$

(i) Show that on all characteristics, $u(t) \to 1$ as $t \to \infty$, independent of the value of $g(x)$.

(ii) Recall from Section 3.4 of Shearer and Levy's book that for the inviscid Burgers' equation, method of characteristics fail if $g'(x) < 0$ for some $x$ and we can identify the time of its failure by studying a PDE for $p := u_x$.

Show that in the forced Burgers' equation, the method of characteristics fails if $g'(x) < -1$ for some $x$.

Suggested Solutions:

(i) Consider a parameterized curve of the form $(t, x(t))$.[4] By writing $u(t,x) = u(t,x(t))$, the total derivative with respect to $t$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t,x(t)) = \frac{\partial u}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial u}{\partial x} \underbrace{=}_{\text{PDE}} 1 - u$$

if we set

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u.$$

Hence, the system of ODEs to be solved is given by

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = u(t), \\ \frac{\mathrm{d}u(t)}{\mathrm{d}t} = 1 - u(t). \end{cases} \tag{24}$$

From the second equation, we can solve using separation of variables to obtain

$$u(t) = 1 - (1 - u(0))e^{-t}. \tag{25}$$

Consider an arbitrary starting point $x_0$ along $t = 0$. Then, $u(0) = u(0, x_0) = g(x_0)$. Thus, along the characteristic curve determined by the ODE for $x(t)$ in (24), from (25), we have

$$u(t, x(t)) = 1 - (1 - g(x_0))e^{-t}.$$

Observe that

$$\lim_{t\to\infty} u(t) = 1$$

independent of the starting point $x_0$ (and hence independent of the characteristic curve of interest).

(ii) Following the method outlined in the previous page, we take the partial derivative with respect to $x$ on the PDE to obtain

$$u_{tx} + (u_x)^2 + uu_{xx} = -u_x.$$

Furthermore, differentiating the initial data with respect to $x$ yields $u_x(0,x) = g'(x)$. Now, let $p := u_x$. Then, $p$ satisfies the following PDE:

$$\begin{cases} p_t + up_x = -p^2 - p & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}, \\ p(0,x) = g'(x) & \text{for } (t,x) \text{ on } \{t=0\} \times \mathbb{R}. \end{cases}$$

Let $p(t,x) = p(t,x(t))$. By the method of characteristics, if we set $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = u(t)$ (which is consistent with (25), implying that $p$ shares the same family of characteristic curves with $u$), then we have the following ODE for $p$:

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = -p(t) - p^2(t). \tag{26}$$

[4] If we see $1 \cdot u_t + \ldots$ in our PDE, it is likely that a parametrized curve with time as a parameter (ie $(t, x(t))$) would work out well too and possibly simplify some of the computations as compared to using $(t(s), x(s))$.

Now, suppose that the characteristic curve starts from some $x_0 \in \mathbb{R}$ at $t = 0$. This implies that along the "common" characteristics, we have

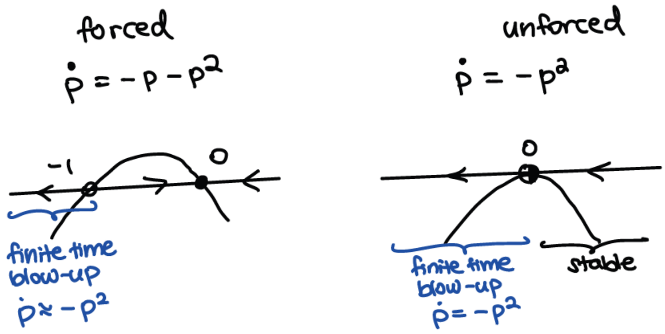
$$p(0) = u_x(0, x(0)) = u_x(0, x_0) = g'(x_0).$$

Henceforth, the question reduces to:

*For what range of initial data $g'(x_0)$ can we solve the ODE in (26) globally in time?*

This is a problem that is more suitable using techniques in nonlinear ODEs (ie Math 266A or Math 134). One method to do this is to solve the ODE by separation of variables and analyze the exact solution. Another method is by using a phase portrait, which we will employ here.

By looking at the phase portrait for the ODE in (26), there are two equilibrium points $p = -1$ and $p = 0$, in which the ODE is approximately linear close to each of these equilibrium points. Since $p = 0$ is a stable point, initial data with $p(0) \in (-1, \infty)$ will approach 0 as $t \to \infty$. On the other hand, since $p = -1$ is an unstable point, with $\frac{\mathrm{d}p}{\mathrm{d}t} \approx -p^2$ for large $p$, we see that for initial data with $p(0) < -1$, it will exhibit finite time blow up[5] (and hence solutions cannot exists globally in time). Hence, a global solution does not exists for $p(0) < -1$, implying that the method of characteristics will fail for the forced inviscid Burgers' equation if there is a point $x$ such that $g'(x) < -1$. $\square$




Remark: It is worth comparing the above argument to that for the (unforced/free) inviscid Burgers' equation. By repeating the arguments in the Shearer and Levy's book, the analogous ODE for $p$ is given by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -p^2.$$

There is only one equilibrium point; the half-stable point (stable from the right) at $p = 0$. Hence, for initial data $p(0) < 0$, the flow is directed away from the equilibrium point towards $-\infty$ with $\frac{\mathrm{d}p}{\mathrm{d}t} = -p^2$, hence exhibiting finite time blow up.

Note that the method of characteristics can be generalized easily to $\mathbb{R}^n$. The example on the next page illustrates this.

[5] A well-known heuristic for nonlinear ODEs of the form $\frac{\mathrm{d}p}{\mathrm{d}t} = p^\alpha$ is that $\alpha < 1 \implies$ lack of uniqueness, while $\alpha > 1 \implies$ potential finite time blow-up.

**Example 5.** (Spring 17, Problem 1(a), Hyperbolic.) Let $\rho : (t, x) \in [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ be an unknown function satisfying the continuity equation:

$$\begin{cases} \rho_t + \nabla \cdot (\vec{v}\rho) = 0 & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^3, \\ \rho(0,x) = \rho_0(x) & \text{for } (t,x) \text{ on } \{t=0\} \times \mathbb{R}^3, \end{cases}$$

with $\vec{v} : x \in \mathbb{R}^3 \to \mathbb{R}^3$.

Represent $\rho$ in terms of $\rho_0$ using the method of characteristics, assuming that $\vec{v}$ is Lipschitz continuous. Explain where the Lipschitz continuity assumption is used in the argument.

Suggested Solutions:

By the product rule, we have

$$\rho_t + \vec{v} \cdot \nabla \rho + (\nabla \cdot \vec{v})\, \rho = 0. \tag{27}$$

Consider a characteristic curve of the form $(t, \mathbf{x}(t))$, with $\mathbf{x}(t) \in \mathbb{R}^3$. By writing $\rho(t, \mathbf{x}) = \rho(t, \mathbf{x}(t))$, the total derivative with respect to $t$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t, \mathbf{x}(t)) = \rho_t + \nabla \rho \cdot \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} \underbrace{=}_{(27)} -(\nabla \cdot \vec{v})\rho$$

if we set

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \vec{v}(\mathbf{x}(t)).$$

Hence, we reduce the continuity equation into the following system of ODEs:

$$\begin{cases} \frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = -(\nabla \cdot \vec{v})(\mathbf{x}(t)) \cdot \rho(t), \\ \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \vec{v}(\mathbf{x}(t)). \end{cases} \tag{28}$$

Since $\vec{v} : \mathbb{R}^3 \to \mathbb{R}^3$ is (globally) Lipschitz continuous, by Picard-Lindelöf theorem, there exists a unique global solution for the ODE for $\mathbf{x}(t)$ in (28) with any initial data. This implies that we have

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \vec{v}(\mathbf{x}(s))\mathrm{d}s. \tag{29}$$

Furthermore, since $\vec{v} : \mathbb{R}^3 \to \mathbb{R}^3$ is Lipschitz continuous, by the Rademacher's theorem[6], it is differentiable almost everywhere. Henceforth, $-(\nabla \cdot \vec{v})(\cdot)$ is Lebesgue integrable on $\mathbb{R}^3$. This implies that we can solve the ODE for $\rho(t)$ in (28) to obtain

$$\begin{aligned} \rho(t, \mathbf{x}(t)) &= e^{-\int_0^t (\nabla \cdot \vec{v})(\mathbf{x}(s))\mathrm{d}s} \rho(0, \mathbf{x}(0)) \\ &= e^{-\int_0^t (\nabla \cdot \vec{v})(\mathbf{x}(s))\mathrm{d}s} \rho_0(\mathbf{x}(0)) \\ &\underbrace{=}_{(29)} e^{-\int_0^t (\nabla \cdot \vec{v})(\mathbf{x}(s))\mathrm{d}s} \rho_0 \left( \mathbf{x}(t) - \int_0^t \vec{v}(\mathbf{x}(s))\mathrm{d}s \right). \end{aligned} \tag{30}$$

Hence, for a given $(t, \mathbf{x})$, we can solve the ODE for $\mathbf{x}(t)$ backwards in time to obtain the value of $\mathbf{x}(0)$ at $t = 0$ as described in (30). Denote the ODE solution as $\Gamma(s; \mathbf{x})$ for each $0 \le s \le t$ with $\Gamma(t; \mathbf{x}) = \mathbf{x}$. By repeating the above arguments but replacing $\mathbf{x}(t)$ with $\Gamma(t; \mathbf{x})$, we have

$$\boxed{\rho(t, \mathbf{x}) = e^{-\int_0^t (\nabla \cdot \vec{v})(\Gamma(s;\mathbf{x}))\mathrm{d}s} \rho_0 \left( \mathbf{x} - \int_0^t \vec{v}(\Gamma(s; \mathbf{x}))\mathrm{d}s \right)}. \tag{31}$$

Remark: Part (ii) of this qual problem (together with part (i)) is available under "qual problems for additional practice:"

[6] Rademacher's theorem says that Lipschitz continuous $\iff$ Differentiable a.e.

## Fully nonlinear first-order PDE and Hamilton-Jacobi Equations.

(Evans Chapter 3.2 and 3.3.)

Let us consider a general nonlinear first-order PDE for the unknown function $u(x) : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\begin{cases} F(\nabla u, u, x) = 0 & \text{for } x \in U, \\ u = g & \text{on } \Gamma, \end{cases}$$

where $\Gamma \subset \partial U, g : \Gamma \to \mathbb{R}$ are given.

To solve this PDE by the method of characteristics, we first consider a curve that is parametrically described by $\mathbf{x}(s) = (x^1(s), \cdots, x^n(s))$, with $s \in I \subset \mathbb{R}$ in some interval $I$. Assuming that $u \in C^2(U)$, we also define

$$z(s) := u(\mathbf{x}(s)) \tag{32}$$

and

$$\mathbf{p}(s) := \nabla u(\mathbf{x}(s)); \quad \text{i.e. } p^i(s) = u_{x_i}(\mathbf{x}(s)) \ \forall i. \tag{33}$$

Here, $\mathbf{p}$ records the derivatives of $u$. Note that this is not done in the method of characteristics for semilinear first-order PDE. The objective remains unchanged - choose the function $\mathbf{x}(\cdot)$ such that we can reduce the PDE into a system of ODEs (which Evans call this the **characteristic ODE**).

Following the derivation in Evans (Section 3.2.1), if we view the PDE as

$$F(\mathbf{p}(s), z(s), \mathbf{x}(s)) = 0,$$

then the characteristic system of ODE is given by

$$\boxed{\begin{cases} \dot{\mathbf{p}}(s) = -\nabla_x F(\mathbf{p}(s), z(s), \mathbf{x}(s)) - \nabla_z F(\mathbf{p}(s), z(s), \mathbf{x}(s))\mathbf{p}(s), \\ \dot{z}(s) = \nabla_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \cdot \mathbf{p}(s), \\ \dot{\mathbf{x}}(s) = \nabla_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)). \end{cases}} \tag{34}$$

Hence, with the right boundary conditions, one can solve any nonlinear first-order PDE by instead solving a system of $2n + 1$ ODEs in (34).

An example of a fully nonlinear first-order PDE on the quals is the Hamilton-Jacobi (HJ) equation. These equations are in the form:

$$\begin{cases} u_t + H(\nabla_x u) = 0 & \text{for } (t, x) \in (0, \infty) \times \mathbb{R}^n, \\ u = g & \text{for } (t, x) \text{ on } \{t = 0\} \times \mathbb{R}^n, \end{cases} \tag{35}$$

Here, $u : [0, \infty) \times \mathbb{R}^n$ is the unknown function with $u(t, x)$, $H : \mathbb{R}^n \to \mathbb{R}$ is the **Hamiltonian**.

Note that solutions to the HJ equation in general are only Lipschitz continuous, so a notion of weak solution is required, with the appropriate assumption. For more information, see Chapter 3.3 in Evans.

To solve the HJ equation, we first define the notion of a Legendre transform as follows:

$$L^*(p) := \sup_{v \in \mathbb{R}^n} \{p \cdot v - L(v)\} \quad \text{for } p \in \mathbb{R}^n.$$

Suppose that:

- The mapping $p \mapsto H(p)$ is convex,
- $\lim_{|p| \to \infty} \frac{H(p)}{|p|} = \infty$,

we define $L := H^*$, known as the **Lagrangian**, which satisifies

- The mapping $p \mapsto L(p)$ is convex,
- $\lim_{|p| \to \infty} \frac{L(p)}{|p|} = \infty$.

Nonetheless, we have the following theorem:

**Theorem.** Let $L$ be the associated Lagrangian to the HJ equation in (35) obtained by taking the Legendre transform of $H$. Then, a weak solution to (35) is given by

$$u(t,x) = \min_{y\in\mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}. \tag{36}$$

Here, (36) is known as the **Hopf-Lax formula**. We will look at an example of utilizing this in the problem below.

**Example 6.** (Spring 21, Problem 8, Elliptic.) The function $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ satisfies the equation:

$$\begin{cases} u_t + \frac{1}{2}(u_x)^2 = 0 & \text{for } (t, x) \in (0, \infty) \times \mathbb{R}, \\ u(0, x) = \frac{1}{2(x^2+1)} & \text{for } (t, x) \text{ on } \{t = 0\} \times \mathbb{R}. \end{cases}$$

(i) Show that solution found by method of characteristics is only valid for a time interval $0 < t < T$. What is the value of $T$?

(ii) Derive a formula for $u(t, 0)$ for all $t > 0$.

Suggested Solutions:

(i) Note that this is a fully nonlinear first-order PDE, so we have to use the formula in (34). To do so, let $z(s) := u(\mathbf{x}(s))$ and $\mathbf{p}(s) := \nabla u(\mathbf{x}(s))$, and see that

$$F(\mathbf{p}(s), z(s), \mathbf{x}(s)) = p^1(s) + \frac{1}{2}\left(p^2(s)\right)^2.$$

Using the formula in (34), we have

$$\begin{cases} \dot{\mathbf{p}}(s) = \mathbf{0}, \\ \dot{z}(s) = \begin{pmatrix} 1 \\ p^2(s) \end{pmatrix} \cdot \begin{pmatrix} p^1(s) \\ p^2(s) \end{pmatrix} = p^1(s) + (p^2(s))^2, \\ \dot{\mathbf{x}}(s) = \begin{pmatrix} 1 \\ p^2(s) \end{pmatrix}. \end{cases}$$

Solving the ODE in $\mathbf{p}(s)$ implies that $p^1(s) = p^1(0)$ and $p^2(s) = p^2(0)$. Plugging this into the ODE for $z(s)$ implies

$$z(s) = z(0) + \left(p^1(0) + \left(p^2(0)\right)^2\right) s. \tag{37}$$

$$z(s) = z(0) + \left(p^1(0) + \left(p^2(0)\right)^2\right) s.$$

Similarly, plugging this into the ODE for $\mathbf{x}(s)$ while setting $t(0) = 0$ implies

$$\begin{pmatrix} t(s) \\ x(s) \end{pmatrix} = \begin{pmatrix} s \\ x(0) + p^2(0)s \end{pmatrix}. \tag{38}$$

$$\begin{pmatrix} t(s) \\ x(s) \end{pmatrix} = \begin{pmatrix} s \\ x(0) + p^2(0)s \end{pmatrix}.$$

Now, let $\begin{pmatrix} t(s) \\ x(s) \end{pmatrix} = \begin{pmatrix} t \\ x \end{pmatrix}$. and $\begin{pmatrix} t(0) \\ x(0) \end{pmatrix} = \begin{pmatrix} 0 \\ x_0 \end{pmatrix}$. Plugging this into (38), we have

$$x = x_0 + p^2(0)t. \tag{39}$$

To determine $p^1(0) = u_t(0, x_0)$ and $p^2(0) = u_x(0, x_0)$, differentiating the initial data with respect to $x$ gives

$$u_x(0, x_0) = -\frac{x_0}{(1 + x_0^2)^2}.$$

Using the PDE, we have

$$u_t(0, x_0) = -\frac{1}{2}(u_x(0, x_0))^2 = -\frac{x_0^2}{(1 + x_0^2)^4}.$$

Plugging the value of $p^2(0)$ into (39), the characteristic curves are given by

$$x = x_0 - \frac{x_0}{(1 + x_0^2)^2}t. \tag{40}$$

Recall that the method of characteristics works up till when two characteristic curves intersect. Hence, let $(x,t)$ be in which two different characteristic curves intersect, characterized by different $x_0$ (ie $x_0$ and $\tilde{x_0}$). Hence, we have

$$\begin{cases} x = x_0 - \frac{x_0}{(1+x_0^2)^2}t, \\ x = \tilde{x_0} - \frac{\tilde{x_0}}{(1+\tilde{x_0}^2)^2}t. \end{cases} \tag{41}$$

Eliminate for $x$ in (41), we have

$$t = \frac{x_0 - \tilde{x_0}}{\frac{x_0}{(1+x_0^2)^2} - \frac{\tilde{x_0}}{(1+\tilde{x_0}^2)^2}}. \tag{42}$$

Iterating over all possible $x_0$ and $\tilde{x_0}$, this implies that two different characteristic curves first intersect at

$$\boxed{T = \inf_{x_0 \neq \tilde{x_0}} \frac{x_0 - \tilde{x_0}}{\frac{x_0}{(1+x_0^2)^2} - \frac{\tilde{x_0}}{(1+\tilde{x_0}^2)^2}}}. \tag{43}$$

(ii) Here, we employ the Hopf-Lax formula as in (36). Since $H(p) = \frac{p^2}{2}$, we can compute its Lagrangian by computing the Legendre transform as follows:

$$L(p) := H^*(p) = \sup_{v\in\mathbb{R}}\{p\cdot v - H(v)\} = \sup_{v\in\mathbb{R}}\{p \cdot v - \frac{v^2}{2}\}.$$

Using standard techniques for unconstrainted optimization in $\mathbb{R}$, we can show that the supremum is obtained when $v = p$, and thus

$$L(p) = p\cdot p - \frac{p^2}{2} = \frac{p^2}{2}.$$

Using the Hopf-Lax formula, we have

$$\begin{aligned} u(t,x) &= \min_{y\in\mathbb{R}}\left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\} \\ u(t,0) &= \min_{y\in\mathbb{R}}\left\{t\frac{\left(\frac{-y}{t}\right)^2}{2} + \frac{1}{2(y^2+1)}\right\} \\ &= \min_{y\in\mathbb{R}}\left\{\frac{y^2}{2t} + \frac{1}{2(y^2+1)}\right\}. \end{aligned} \tag{44}$$

To optimize the function $f(y;t) = \frac{y^2}{2t} + \frac{1}{2(y^2+1)}$, we take the derivative with respect to $y$ to obtain

$$\frac{\partial f}{\partial y}(y;t) = \frac{y}{t} - \frac{y}{(1+y^2)^2}.$$

The stationary point(s) are

$$\begin{cases} y = 0, \quad 1+y^2 = \sqrt{t}, & \text{if } t > 1, \\ y = 0, & \text{if } 0 < t \leq 1. \end{cases} \tag{45}$$

For $t > 1$, observe that $y = 0$ corresponds to a local maximum, while the two values of $y$ as solutions to $1 + y^2 = \sqrt{t}$ corresponds to a symmetric local minimum. Hence, the minimizer is given by the values of $y$ for which $1 + y^2 = \sqrt{t}$.

For $0 < t \leq 1$, $y = 0$ is the only stationary point and is a local minimizer.

Combining these facts, we thus have

$$\boxed{u(t,0) = \begin{cases} \frac{1}{2} & \text{for } 0 < t \leq 1, \\ \frac{\sqrt{t}-1}{2t} + \frac{1}{2\sqrt{t}} & \text{for } t > 1. \end{cases}} \tag{46}$$

Qual problems for additional practice:

**Exercise 2.** (Spring 15, Problem 3, Hyperbolic.)
Solve for $\mathbf{u} : [0,\infty) \times \mathbb{R}^2$ that satisfies

$$\frac{\partial u_i}{\partial t}(t,\mathbf{x}) + \sum_{j=1}^{2} \frac{\partial u_i}{\partial x_j}(t,\mathbf{x}) u_j(t,\mathbf{x}) = -u_i(t,\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^2, t \in (0,\infty), \text{ for each } i$$

with initial conditions

$$\mathbf{u}(0,\mathbf{x}) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \text{ where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

**Exercise 3.** (Spring 17, Problem 1, Hyperbolic.) Let $\rho : (t,x) \in [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ be an unknown function satisfying the continuity equation:

$$\begin{cases} \rho_t + \nabla \cdot (\vec{v}\rho) = 0 & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^3, \\ \rho(0,x) = \rho_0(x) & \text{for } (t,x) \text{ on } \{t=0\} \times \mathbb{R}^3, \end{cases}$$

with $\vec{v} : x \in \mathbb{R}^3 \to \mathbb{R}^3$.

(i) Represent $\rho$ in terms of $\rho_0$ using the method of characteristics, assuming that $\vec{v}$ is Lipschitz continuous. Explain where the Lipschitz continuity assumption is used in the argument.

(ii) Suppose that $\nabla \cdot \mathbf{v} > -1$ in $\mathbb{R}^3$ and $\rho_0 = \chi_{\{|x|<1\}}$, where $\chi_A$ denotes the characteristic function of a set $A$. Show that $\Omega_1 := \{x : \rho(1,x) > 0\}$ has volume greater than $4/3$.
(Hint: You may use the fact, which follows from your answer for (i), that the solution of $u_t + \nabla u \cdot \vec{v} = 0$ shares the same characteristic path as $\rho$.)

**Exercise 4.** (Spring 16, Problem 4, Elliptic.)
Solve the Hamilton-Jacobi equation

$$\phi_t + |\phi_x| = 0, \quad x \in \mathbb{R}, \ t > 0$$

with initial data

$$\phi(0,x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x > 0. \end{cases}$$

**Exercise 5.** (Fall 16, Problem 4, Elliptic.)
Consider $\phi(t,x) : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ solving the Hamilton-Jacobi equation

$$\phi_t + |\nabla \phi| = 0, \quad x \in \mathbb{R}^n, \ t > 0$$

with initial data

$$\phi(0,x) = \max\{|x|^2 - 1, 0\}.$$

Show that $\phi(x,t) = 0$ when $t = |x| - 1$.

**Exercise 6.** (Spring 23, Problem 5, Elliptic.) Consider the one-dimensional PDE

$$\begin{cases} u_t + \frac{1}{2}(u_x)^2 = 0 & \text{for } (t,x) \in [0,\infty) \times \mathbb{R}, \\ u(0,x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases} & \text{for } (t,x) \text{ on } \{t=0\} \times \mathbb{R}, \\ u(t,x) \to 0 \text{ as } x \to -\infty, \text{ and} \\ u(t,x) \to 1 \text{ as } x \to \infty. \end{cases}$$

(i) Show that the PDE does *not* have a travelling-wave solution that is compatible with these boundary conditions, even when the derivatives are interpreted in the sense of distributions.

(ii) Derive the weak solution to the PDE.
(Note: You do not need to derive the Hopf–Lax formula, but you should state the formula carefully if you use it.)

# 3 Discussion 3

Fourier Series and Separation of Variables.
(References: Shearer and Levy Chapters 6 and 7).

For this discussion, the emphasis is on solving boundary value problems, i.e, considering a finite domain in space such that we impose certain values on the unknown function $u$ and/or its derivative $u_x$ on the boundary of the aforementioned space. In other words, we will be solving PDE problems (i.e say diffusion equation) that look like this

$$\begin{cases} u_t(x,t) = \nu u_{xx}(x,t) & \text{in } (a,b)\times(0,\infty), \\ u(a,t) = f(t) & \text{on } \{x=a\}\times[0,\infty), \\ u(b,t) = g(t) & \text{on } \{x=b\}\times[0,\infty), \\ u(x,0) = \phi(x) & \text{on } [a,b]\times\{t=0\}. \end{cases} \tag{47}$$

These PDEs are amenable to a technique known as the separation of variables, in which one looks for a solution of the form $u(x,t) = X(x)T(t)$ for some functions $X$ and $T$. More details on this will be provided in the example below.

Before we try to solve some of these, here are some results covered in 266A that will be useful:

- For an ODE for $y(x)$ of the form $(py')' + qy = -\lambda wy$, with $p, q, w$ as functions such that $p, q, w$, and $p'$ are continuous on $[a,b]$, $p(x), w(x) > 0$ for all $x \in [a,b]$, the problem has a separated boundary conditions of the form:
$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \alpha_1, \alpha_2 \text{ not both } 0,$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0, \quad \beta_1, \beta_2 \text{ not both } 0,$$
then we say that it is in **Sturm-Liouville** form. This implies that the eigenvalues $\lambda_1, \cdots$ are real with $\lambda_n \to \infty$ as $n \to \infty$, and the normalized eigenfunctions forms an orthonormal basis under the $w$-weighted inner product $(f,g)_w = \int_a^b f(x)g(x)w(x)\mathrm{d}x$ for $L^2([a,b])$.
- The set $\{\cos\left(\frac{n\pi x}{L}\right)\}_{n=0}^{\infty} \cup \{\sin\left(\frac{n\pi x}{L}\right)\}_{n=1}^{\infty}$ forms an orthonormal basis under the usual inner product $(f,g) = \int_a^b f(x)g(x)\mathrm{d}x$ for $L^2([-L,L])$ for any $L > 0$.
- This implies that for any $f \in L^2([-L,L])$, we can express it as its **Fourier** series as follows:
$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$
with
$$a_n = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) f(x) \quad \text{for } n \geq 0,$$
and
$$b_n = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) f(x) \quad \text{for } n \geq 1.$$
- The Fourier series always converges in $L^2$ to the function that is trying to represent, pointwise at every $x$ such that $f(x)$ is continuous, and converges to $\frac{1}{2}(f(x^-) + f(x^+))$ at every $x$ for which $f(x)$ is discontinuous.
- If a function is only defined on $[0,L]$, we can consider the even $f_e(x)$ and odd $f_o(x)$ extension of the function to $[-L,L]$. Since $f_e(x)$ is even, we have $b_n = 0$ and $a_n = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) f(x)\mathrm{d}x$, and the corresponding series representing is thus valid on the original domain of $f$, on $[0,L]$. Similarly, since $f_o(x)$ is odd, we have $a_n = 0$ and $b_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\mathrm{d}x$.

With that, we are ready to solve a PDE by separation of variables as follows.

**Example 7.** (Spring 17, Problem 5, Hyperbolic.) Consider the following PDE

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) & \text{in } (0,1)\times(0,\infty), \\ u_x(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ u_x(1,t) = 0 & \text{on } \{x=1\}\times[0,\infty), \\ u(x,0) = (s-1)x & \text{on } [0,1]\times\{t=0\}, \\ u_t(x,0) = 0 & \text{on } [0,1]\times\{t=0\}, \end{cases} \tag{48}$$

for a constant $s \in \mathbb{R}$.

(i)  Solve the PDE.[a]

(ii) Define

$$e(t) = \int_0^1 (u_t(x,t))^2 + (u_x(x,t))^2 \mathrm{d}x.$$

Show that $e(t) = (s-1)^2$.

[a] If you look at the actual problem, it gives a hint to ask you to consider an even extension of the initial data. However, if you are solving the PDE by separation of variables, this is not necessary or required.

Suggested Solutions:

(i) Here, we will attempt to solve this by separation of variables as follows.

Step 1: Look for separable solutions and derive boundary conditions.

Here, we look for **non-trivial** solutions[7] of the form

$$u(x,t) = X(x)T(t)$$

(ie separable). Using the Neumann boundary conditions, this implies that

$$u_x(0,t) = X'(0)T(t) = 0, \quad \text{and } u_x(1,t) = X'(1)T(t) = 0. \tag{49}$$

From the first equation, either $X'(0) = 0$ or $T(t) = 0$ for all $t \geq 0$. However, the latter implies that $u(x,t) = X(x)T(t) = 0$ since $T$ is now the zero function, and we obtain a trivial (zero) "solution" to the above PDE, which clearly does not satisfy $u(x,0) = (s-1)x$. Similarly, one deduces that $X'(1) = 0$. In summary,

$$X'(0) = X'(1) = 0. \tag{50}$$

Plugging this into the PDE, we get

$$\begin{aligned} u_{tt}(x,t) &= X(x)T''(t) \\ u_{xx}(x,t) &= X''(x)T(t) \\ u_{tt}(x,t) - u_{xx}(x,t) &= X(x)T''(t) - X''(x)T(t) = 0. \end{aligned} \tag{51}$$

Dividing both sides of the equation by $X(x)T(t)$,[8] we obtain

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} \tag{52}$$

[7] "Is zero a solution" can be easily checked by substituting it into the PDE such that it satisfies the initial condition. Thus, it makes sense to just search for non-zero solutions.

[8] We did not discuss about the possibility of $X$ and $T$ being $0$ at a point. Most books, not even Shearer and Levy, discuss this. The most convincing argument I have for you (at least, I am convinced) is that we use the argument in (53) as an intuition, and then come back to (52) and postulate that $X(x)$ are solutions to the eigenvalue problem $X''(x) = -\lambda X(x)$ for $\lambda$ independent of $x$ and $t$. Note that philosophically, this makes sense because when we write an ansatz/guess to the PDE, we are already restricting the functions that we are looking for to a smaller space. If it ends up not working, then it implies that either the restriction is too restrictive (say you have an ansatz $u(x,t) = 1$) or there really is no solution. Substitute this into (51), we obtain $X(x)T''(t) = -\lambda T(t)X(x)$. Since this holds for all $x$ and $t$, we pick a point $x^*$ such that $X(x^*) \neq 0$, and then divide by $X(x^*)$ on both sides to obtain $T''(t) = -\lambda T(t)$. If such a point does not exists, this implies that $X(x) = 0$ for all $x$, and thus $u(x,t) = X(x)T(t)$, the trivial solution.

Since the LHS of (52) only depends on $x$, and the RHS of (52) only depends on $t$, then (52) is equals to a constant. One way to understand this is that if $x$ varies while keeping $t$ fixed, it does not change the value on the right. This implies that it must be a constant in $x$ for any given $t$. Using a similar argument, we then have that it is a constant in $t$ for any given $x$.[9] Thus, (52) becomes

$$-\frac{X''(x)}{X(x)} = -\frac{T''(t)}{T(t)} = \lambda \tag{53}$$

where $\lambda$ is a constant. In fact, this constant must be a real, since $\lambda$ here is viewed as the eigenvalue to the problem $-X''(x) = \lambda X(x)$ with boundary terms $X'(0) = X'(1) = 0$. This makes it a Strum-Liouville problem and hence, the constant $\lambda$ must be real.

Step 2: Solve the corresponding eigenvalue problem in $X$.

Now, we would like to solve the eigenvalue problem

$$\begin{cases} -X''(x) = \lambda X(x) \\ X'(0) = X'(1) = 0 \end{cases}$$

to obtain the corresponding eigenvalues and more importantly, eigenfunctions. If $\lambda = 0$, then $X(x) = Ax + B$. Using $X'(0) = X'(1) = 0$, we can only determine that $A = 0$. Thus, $X(x) = B$, an arbitrary constant, is an eigenfunction. In particular, the function $1$ is an eigenfunction.

For $\lambda < 0$, the general solution is given by

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}.$$

Plugging the initial conditions in gives $X(x) \equiv 0$, which is not what we are looking for.

For $\lambda > 0$, The general solution is given by

$$\begin{aligned} X(x) &= A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \\ X'(x) &= A\sqrt{\lambda}\cos(\sqrt{\lambda}x) + B\sqrt{\lambda}\sin(\sqrt{\lambda}x). \end{aligned}$$

Using $X'(0) = 0$, this implies that $A\sqrt{\lambda} = 0$. Since $\lambda > 0$, this implies that $A = 0$. With $X'(x) = B\sqrt{\lambda}\sin(\sqrt{\lambda}x)$ left, we use the condition $X'(1) = 0$ to obtain

$$X'(L) = B\sqrt{\lambda}\sin(\sqrt{\lambda}) = 0.$$

Note that it is now possible for this expression to be $0$ with $B \neq 0$. This happens when $\sin(\sqrt{\lambda}) = 0$, or when $\sqrt{\lambda} = n\pi$ for $n \in \mathbb{N} \setminus \{0\}$, or

$$\lambda_n = n^2\pi^2, \quad n \in \mathbb{N} \setminus \{0\}.$$

The corresponding eigenfunctions (the functions attached to $B$ in $X(x)$ since $B \neq 0$, and combining with the case when $\lambda > 0$) are

$$X_n(x) = \cos(\sqrt{\lambda_n}) = \cos(n\pi x), \quad n \in \mathbb{N},$$

since these functions will satisfy the boundary conditions but are non-zero functions.[10]

Step 3: Solve the corresponding ODE in $T$.[11]

Going back to (53), this implies that there are only countably finitely many $\lambda$ (given by $\lambda_n$ above) that gives a non-zero solution. Thus, for each $n \in \mathbb{N}$, we will be solving the ODE:

$$-T_n''(t) = \lambda_n T_n(t),$$

[9] If you don't buy this argument, take $\lambda(x,t) = \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}$. Note that $\frac{\partial}{\partial x}\lambda(x,t) = 0$ since $\frac{\partial}{\partial x}\lambda(x,t) = \frac{\partial}{\partial x}\frac{T''(t)}{T(t)} = 0$ is independent of $t$. Similarly, $\frac{\partial}{\partial t}\lambda(x,t) = \frac{\partial}{\partial t}\frac{X''(t)}{X(t)} = 0$. Thus, $\lambda_x = \lambda_t = 0$ implies that $\lambda$ is a constant.

[10] As seen from the arguments here, there are countably many eigenvalues (and hence eigenfunctions). This is consistent with the Strum-Liouville theory.

[11] Note: This is **not** an eigenvalue problem for the simple reason that no boundary conditions were given for $T(t)$. You can't deduce that $T(t)$ is some function in $t$ using the initial condition since they are not necessarily the zero initial condition, so a similar argument in Step 1 does not hold.

where we index the function $T(t)$ by $n$ to imply that we are solving a different ODE for different $n$ (due to different values of $\lambda_n$). Since $\lambda_n \geq 0$, for $\lambda = 0$ (ie at $n = 0$), we obtain

$$T_0(t) = A_0 t + \frac{B_0}{2}.$$

Recall that for each $n$, we are solving a different ODE, so the arbitrary constants are different, and thus are indexed by $n$.
For $\lambda_n > 0$ (ie for $n \geq 1$), we obtain

$$\begin{aligned} T_n(t) &= A_n \sin(\sqrt{\lambda_n} t) + B_n \cos(\sqrt{\lambda_n} t) \\ &= A_n \sin(n\pi t) + B_n \cos(n\pi t). \end{aligned}$$

Step 4: Obtain general solution by linearity.

By linearity, for each $n$, the solution $u_n(x,t) = X_n(x)T_n(t)$ is a solution. Thus, a linear combination of these $u_n(x,t)$ is also a solution. This implies that[12]

$$\begin{aligned} u(x,t) &= \sum_{n\in\mathbb{N}} u_n(x,t) \\ &= \sum_{n\in\mathbb{N}} X_n(x)T_n(t) \\ &= (1)\left(A_0 t + \frac{B_0}{2}\right) + \sum_{n\in\mathbb{N}, n\geq 1} \cos(n\pi x)\left(A_n \sin(n\pi t) + B_n \cos(n\pi t)\right). \end{aligned}$$

Step 5: Solve for the "Fourier" coefficients.

First, see that (since $\sin(0) = 0, \cos(0) = 1$),

$$(s-1)x = u(x,0) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \cos(n\pi x) = \sum_{n=0}^{\infty} B_n \cos(n\pi x).$$

Thus, the $B_n$ are coefficients of the Fourier cosine series. As explained before, the coefficients are given by

$$B_n = \frac{2}{L}\int_0^L \cos(n\pi x)\,(s-1)x \,\mathrm{d}x, \quad n \geq 0. \tag{54}$$

Next, take $\frac{\partial}{\partial t}$ to obtain

$$\begin{aligned} u_t(x,t) &= A_0 + \sum_{n\in\mathbb{N}, n\geq 1} \cos(n\pi x)\left(n\pi A_n \cos(n\pi t) - n\pi B_n \sin(n\pi t)\right) \\ 0 = u_t(x,0) &= A_0 + \sum_{n=1}^{\infty} n\pi A_n \cos(n\pi x). \end{aligned}$$

By computing the Fourier cosine series coefficients or noting that a Fourier series is unique, we can see that $A_0 = 0$ and $A_n = 0$ for all $n \geq 1$.

Thus, we have

$$\boxed{u(x,t) = \frac{B_0}{2} + \sum_{n\in\mathbb{N}, n\geq 1} B_n \cos(n\pi x)\, B_n \cos(n\pi t)}.$$

with coefficients $B_n$ given in (54).

[12] We could have written the solution as $C_0(1)(A_0 t + B_0) + \sum_{n\in\mathbb{N}, n\geq 1} C_n \cos(n\pi x)\left(A_n \sin(n\pi t) + B_n \cos(n\pi t)\right)$ with arbitrary constants $C_n$ in front, but these are absorbed in the $A_n$'s and $B_n$'s, so it doesn't really matter.

(ii) We do so by first considering $e'(t)$, which we proceed as follows:

$$
\begin{aligned}
e'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (u_t(x,t))^2 + (u_x(x,t))^2 \mathrm{d}x \\
&= \int_0^1 \frac{\partial}{\partial t}\left((u_t(x,t))^2 + (u_x(x,t))^2\right)\mathrm{d}x \\
&= \int_0^1 2u_t(x,t)u_{tt}(x,t) + 2u_x(x,t)u_{xt}(x,t)\mathrm{d}x \\
&\underbrace{=}_{(48)} \int_0^1 2u_t(x,t)u_{xx}(x,t) + 2u_x(x,t)u_{xt}(x,t)\mathrm{d}x \\
&\underbrace{=}_{\text{IBP}} \int_0^1 \underbrace{2u_t(x,t)u_{xx}(x,t) - 2u_{xx}(x,t)u_t(x,t)}_{\text{cancels each other out}}\mathrm{d}x + \underbrace{2u_x(1,t)u_t(1,t)}_{u_x(1,t)\equiv 0} - \underbrace{2u_x(0,t)u_t(0,t)}_{u_x(0,t)\equiv 0} \\
&= 0.
\end{aligned}
$$

This implies that the energy $e(t)$ is constant in time. Hence, we have

$$
e(t) = e(0) = \int_0^1 (u_t(x,t))^2 + (u_x(x,t))^2 \mathrm{d}x = \int_0^1 0^2 + (s-1)^2 \mathrm{d}x = (s-1)^2.
$$

Energy Estimates/Methods and Grönwall's Inequality/Lemma.

One key tool that we will repeatedly use to prove uniqueness to solutions to parabolic and hyperbolic PDEs for $t \geq 0$ would be the energy estimates. The idea here is to define an energy $E(t)$ that consists of spatial integrals of (derivatives) of the unknown function and is a non-negative function for all time $t > 0$. Such a choice of energy usually comprises of squared terms to ensure that it is non-negative (and these terms might be physically motivated; ie kinetic and/or potential energy). In most cases, one would have to "estimate" the energy at time $t$ (and hence the name of this method), and this can be done as follow:

- Show that the energy is non-increasing, i.e, ($\frac{\mathrm{d}E(t)}{\mathrm{d}t} \leq 0$ for all $t$). This implies that
$$E(t) \leq E(0) \quad \text{for all } t > 0.$$
- Show that the energy is constant in time, i.e, ($\frac{\mathrm{d}E(t)}{\mathrm{d}t} = 0$ for all $t$). This implies that
$$E(t) = E(0) \quad \text{for all } t > 0.$$
- Show that the derivative of the energy has an upper bound that depends on the energy. This would require solving the a differential inequality to obtain an estimate (upper bound) for the energy at time $t$.

In the last case, one often relies on the Grönwall's Inequality to do so. We recall the inequality from 266A as follows:

**Lemma. (Grönwall's Inequality.)** Let $I = [a, b]$, with $a < b$.

- If $u$ and $\beta$ are continuous on $I \subset \mathbb{R}$, $u$ differentiable on $I^\circ$, and
$$u'(t) \leq \beta(t)u(t) \quad \text{for all } t \in I^\circ$$
then
$$u(t) \leq u(a) \exp\left(\int_a^t \beta(s)\mathrm{d}s\right) \quad \text{for all } t \in I.$$
- If $u$ and $\beta$ are continuous on $I \subset \mathbb{R}$, $\alpha$ is integrable on every closed subset of $I$, $\beta(t) \geq 0$, $\alpha(t)$ is non-decreasing in $t$, and
$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)\mathrm{d}s,$$
then
$$u(t) \leq \alpha(t) \exp\left(\int_a^t \beta(s)\mathrm{d}s\right).$$
- If $u$ and $\beta$ are continuous on $I \subset \mathbb{R}$, $\alpha$ is integrable on every closed subset of $I$, $\beta(t) \geq 0$, and
$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)\mathrm{d}s,$$
then
$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(r)\mathrm{d}r\right)\mathrm{d}s.$$

We will see an example below, and more applications of it in the next discussion.

**Example 8.** (Spring 20, Problem 4, Parabolic, Modified) Let $u(x,t)$ a $C^{2,1}$ solution to the equation

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u^2(x,t) - u(x,t) & \text{in } U \times (0,\infty), \\ u(\cdot,t)|_{\partial U} = 0 & \text{on } \partial U \times [0,\infty), \\ u(x,0) = g(x) & \text{on } U \times \{t=0\}, \end{cases} \tag{55}$$

where $U$ is a bounded open domain, and its boundary $\partial U$ is piecewise smooth. Assume that $0 \le g(x) \le \kappa < 1$. We will cover in a future discussion session on how to use the maximum principle to prove that

$$0 \le u(x,t) \le \kappa \tag{56}$$

for all $(x,t) \in U \times [0,\infty)$. Using this fact, prove that

$$\int_U u(x,t)^2 \mathrm{d}x \le \left(\int_U g(x)^2 \mathrm{d}x\right) e^{2(\kappa-1)t}.$$

Suggested Solutions: Using a similar strategy, we define

$$e(t) = \int_U u(x,t)^2 \mathrm{d}x.$$

Next, we compute $e'(t)$ as follows:[13]

$$\begin{aligned} e'(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_U u(x,t)^2 \mathrm{d}x \\ &= \int_U 2u(x,t)u_t(x,t)\mathrm{d}x \\ &\underbrace{=}_{(55)} \int_U 2u(x,t)\left(\Delta u(x,t) + u^2(x,t) - u(x,t)\right)\mathrm{d}x \\ &= \int_U 2u(x,t)\Delta u(x,t) + 2(u(x,t)-1)u^2(x,t)\mathrm{d}x, \\ &\underbrace{=}_{\text{IBP}} \underbrace{-\int_U 2\nabla u(x,t)\cdot\nabla u(x,t)\mathrm{d}x}_{=-2\int_U \|\nabla u\|^2(x,t)\mathrm{d}x \le 0} + \underbrace{\int_{\partial U} 2u(x,t)\nabla u(x,t)\cdot \nu \mathrm{d}S(x)}_{u|_{\partial U}=0} + \int_U 2\underbrace{(u(x,t)-1)}_{\le \kappa-1} u(x,t)^2 \mathrm{d}x \\ &\le 2(\kappa-1)\int_U u(x,t)^2\mathrm{d}x \\ &\le 2(\kappa-1)e(t). \end{aligned}$$

By Grönwall's Inequality, we have

$$e(t) \le e(0)e^{2(\kappa-1)t} = \left(\int_U u(0,x)^2\mathrm{d}x\right)e^{2(\kappa-1)t} = \left(\int_U g(x)^2\mathrm{d}x\right)e^{2(\kappa-1)t}.$$

[13] Note that in the computations below, "IBP" here means integration by parts, though it is more commonly known as the divergence theorem since we have $x \in \mathbb{R}^n$ (rather than just $\mathbb{R}$).

Qual problems for additional practice:

**Exercise 7.** (Fall 18, Problem 8, Hyperbolic.)
The equation of motion of a vibrating beam is

$$-c_m u_{tt} = EI u_{xxxx},$$

where $u$ is the displacement of the beam as a function of position along its axis, the constant $c_m = \rho A$ is the linear mass density of the beam, $E$ is the elastic modulus, and $I$ is the moment of inertia. If the beam is simply supported at the ends, it satisfies the boundary conditions $u(0,t) = u(L,t) = 0$ (no displacement at the ends), and $u_{xx}(0,t) = u_{xx}(L,t) = 0$ (zero bending moments).

(i) Compute the solution of this problem, given the initial displacement $u(x,0) = f(x)$ and initial velocity $u_t(x,0) = g(x)$.

(ii) Find the solution of the vibrating-string equation

$$u_{tt} = c^2 u_{xx}$$

with fixed boundary conditions $u(0,t) = u(L,t) = 0$ and initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$. Compare how the spectrum of the normal modes scales with the length of the string versus the length of the beam.

**Exercise 8.** (Fall 19, Problem 5, Hyperbolic.) Consider functions $v, f : [a,b] \times \mathbb{R}^+ \to \mathbb{R}$ that satisfies

$$\begin{cases} \rho \dfrac{\partial v(x,t)}{\partial t} = \dfrac{\partial p(x,t)}{\partial x} & \text{in } (a,b)\times(0,\infty), \\ p(x,t) = k(f(x,t)-1), & \text{in } (a,b)\times(0,\infty), \\ \dfrac{\partial f}{\partial t}(x,t) = \dfrac{\partial v}{\partial x}(x,t) + \dfrac{1}{W}(1-f(x,t)), & \text{in } (a,b)\times(0,\infty), \\ p(a,t) = 0 & \text{on } \{x=a\}\times(0,\infty), \\ p(b,t) = 0 & \text{on } \{x=b\}\times(0,\infty), \\ v(x,0) = 0 & \text{on } (a,b)\times\{t=0\}, \\ f(x,0) = s & \text{on } (a,b)\times\{t=0\}. \end{cases}$$

(i) Show that $e'(t) \le 0$, with

$$e(t) = \int_a^b \rho \frac{v^2(x,t)}{2} + \frac{k}{2}(f(x,t)-1)^2 \mathrm{d}x$$

for positive constants $\rho$, $k$, and $W$.

(ii) Solve the PDE above when $\frac{1}{W} \equiv 0$.

**Exercise 9.** (Fall 21, Problem 3, Parabolic.) The concentration $c(r,t)$ of diffusing molecules within a droplet obeys the following PDE:

$$\frac{\partial c}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right), \quad r < 2, t > 0,$$

along with the side conditions

$$c(r,0) = \begin{cases} 0 & \text{if } r < 1 \\ 1 & \text{if } r > 1 \end{cases} \qquad \text{and} \qquad -\frac{\partial c}{\partial r}|_{r=2} = 0.$$

Here $r$ is the usual spherical polar coordinate, representing distance to the center of the droplet. By seeking solutions of the form $c(r,t) = \frac{\rho(r)}{r}T(t)$, or otherwise, prove the following facts:

(i) In the limit $t \to \infty$, it is the case that $c(r,t) \to c_\infty$. You should find the constant $c_\infty$.

(ii) Also, in the limit $t \to \infty$, it is the case that $|c(r,t) - c_\infty| < Ce^{-\alpha t}$ for some positive constants $C$ and $\alpha$. You should identify the largest possible value for the constant $\alpha$.

Note: You do not need to give the value of $\alpha$ explicitly; it is sufficient to derive it implicitly as a root of an equation. There is no need to find a value for $C$.

**Exercise 10.** (Spring 15, Problem 5, Parabolic, Modified.) Consider the following PDE

$$u_t - \Delta u + u^2 = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

Suppose that $u$ and $v$ are smooth solutions with $|u|, |v|, \|\nabla u\|, \|\nabla v\| \to \infty$ as $|x| \to \infty$, and are uniformly bounded, i.e there exists a positive constant $M$ such that $|u(x,t)|, |v(x,t)| \leq M$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$. Prove the following stability estimate:

$$\int_{\mathbb{R}^n} |u(x,t) - v(x,t)|^2 \mathrm{d}x \leq \left(\int_{\mathbb{R}^n} |u(x,0) - v(x,0)|^2 \mathrm{d}x\right) e^{2Mt} \quad \text{for all } t > 0.$$

Hint: Note that $u^2 - v^2 \leq |u+v||u-v| \leq 2M|u-v|$.

# 4 Discussion 4

Wave Equation.
(References: Shearer and Levy Chapter 4, Evans Chapter 2.4.)

The following are some key concepts that you should know for wave equation for this class (and for quals too):

- **Solution to the 1D-wave equation.**
  If $u(x,t)$ for $x \in \mathbb{R}$ satisfies the 1D-wave equation given by

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R} \times \{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } \mathbb{R} \times \{t=0\}, \end{cases} \tag{57}$$

  then the solution (also known as the d'Alembert's solution) is given by

$$\boxed{u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\mathrm{d}y.}. \tag{58}$$

  Regularity of $u(x,t)$ is inherited from the initial data $\phi$ and $\psi$. Thus, we can have that $u \in C^{2,2}$ (and hence the second derivatives make sense in the classical manner) if $\phi$ is $C^2$ and $\psi$ is $C^1$. Furthermore,

$$\phi \in C^n \text{ and } \psi \in C^{n-1} \quad \text{for } n \geq 2 \implies u \in C^{n,n}.$$

  In general, if the differential operator is factorizable into two distinct transport operators of the form $(\partial_t + a\partial_x)$, one can repeat the derivative for the solution to (57) above for generic initial data and quarter-plane problems. See Shearer and Levy Chapter 4.2 for more information. The first part of Example 9 covers an example of this.

- **Domain of Dependence and Region of Influence for Wave Equation in $\mathbb{R}^n$.**
  For $n = 1$, we can leverage on the exact solution in (58). It says that the initial data $\phi(x)$ splits into two waves of equal amplitude emanating from its support at $t = 0$. One can draw parallel to this by the method of characteristics, with characteristic curves travelling with velocities $-c$ and $c$. On the other hand, the initial data $\psi(x)$ has its influence spread uniformly over a conical area in spacetime, as illustrated by the integral $\frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\mathrm{d}y$. For more information on the graphical interpretation of this, see Shearer and Levy Chapter 4.2. For an example of this in action, see Example 11.

  For $n \geq 2$, this is covered in Evans Chapter 2.4 and Shearer and Levy Chapter 4.5. Here, we are looking at the wave equation in $\mathbb{R}^n$, given by

$$\begin{cases} u_{tt}(x,t) = c^2 \Delta u(x,t) & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = f(x) & \text{on } \mathbb{R}^3 \times \{t=0\}, \\ u_t(x,0) = g(x) & \text{on } \mathbb{R}^3 \times \{t=0\}. \end{cases} \tag{59}$$

  In general, one can derive the representative formulas for the solution for $n \geq 2$. The conclusion is what known as the **Huygen's principle**, which says that for $n \geq 2$,

  *A "disturbance" originating at $x$ propagates along a sharp wavefront in odd dimensions, while it continues to have effects even after the leading edge of the wavefront pass in even dimensions.*

  Note that one can draw parallel of this to the $n = 1$ case, in which "sharp wavefront" $\approx$ characteristics from the support of $\phi$, and "effects even after the leading edge of the wavefront pass" $\approx$ spreading of influence in a conical manner from the support of $\psi$.

  If you are not allowed to quote this result (ie as required by the problem, or that it is a modified wave equation), then you will have to prove it. Thankfully, there is an easier manner that one can prove an analogous domain of dependence/influence result using the energy method, in which we will cover below. An example of how this is done is covered in Example 10.

- **Uniqueness of Solutions by the Energy Method.**
  For a typical wave equation in $\mathbb{R}^n$ of the form (59), one can define the **energy** as

$$E(t) = \int_{\mathbb{R}^n} |u_t|^2(x,t) + \|\nabla u\|^2(x,t)\mathrm{d}x,$$

where $\nabla$ is to be understood as the spatial derivative, and $\|\cdot\|$ refers to the $l^2$ norm. One can then proceed to prove uniqueness via the following steps:

1. Let $u, v$ be two solutions to (59). Define $w := u - v$ and show that $w$ satisfies the wave equation with the zero initial data given by

$$\begin{cases} w_{tt}(x,t) = c^2 \Delta w(x,t) & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = 0 & \text{on } \mathbb{R}^3 \times \{t = 0\}, \\ u_t(x,0) = 0 & \text{on } \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

2. Show that $E'(t) \leq 0$. This step usually involves applying the wave equation and applying integration by parts (ie divergence theorem). Note that if it is a initial-boundary value problem or a solution with compact support, we usually have that the boundary terms from integration by parts vanish. See the second part of Example 9 for an instance of this.

   However, for a full space problem, this is not guaranteed. Hence, we will have to prove (or cite relevant results to show) that the solution $w$ has compact support. The typical way to do so is to use an energy method (yes, there is an energy method within an energy method) for the local energy to show that the waves have finite speed of propagation, and thus a zero initial data will lead to a zero solution for any time $t > 0$. This usually relies on applyng the **Reynolds Transport Theorem**, (see Appendix C4 of Evans) which we will state below:

   **Proposition.** (Reynolds Transport Theorem.) Consider a family of smooth, bounded regions $U(t) \subset \mathbb{R}^n$ that depends smoothly upon the parameter $t \in \mathbb{R}$. Le $\mathbf{v}$ be the velocity of the moving boundary $\partial U(t)$ and $\hat{\mathbf{n}}$ be the output pointing unit normal. If $f(x,t)$ is a smooth function, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{U(t)} f(x,t) \mathrm{d}x = \int_{U(t)} f_t(x,t) \mathrm{d}x + \int_{\partial U(t)} f(x,t)(\mathbf{v} \cdot \hat{\mathbf{n}})(x,t) \mathrm{d}S(x).$$

   An example of how this is done is covered in Example 10.

3. Hence $0 \leq E(t) \leq E(0) = 0$ (one can compute that $E(0) = 0$). Use the fact that $E(t) = 0$ and $E(t)$ is the integral of a bunch of non-negative terms to deduce that $w_t \equiv w_x \equiv 0$ and hence $w \equiv 0$. This then implies that $u = v$.

Note that this list is not exhaustive, and one should also be familiar with other concepts such as the Duhamel's principle (for solving the forced wave equation; see Shearer and Levy Chapter 4.4).

**Example 9.** (Spring 19, Problem 1, Hyperbolic, Modified.) Let $u(x,t)$ solve the initial value problem:

$$\begin{cases} u_{tt} + u_{xt} - 2u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R} \times \{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } \mathbb{R} \times \{t=0\}, \end{cases} \tag{60}$$

(i) Derive a formula for $u$ in terms of $\phi$ and $\psi$, when $\phi, \psi$ are $C^2$.

(ii) Next, consider the boundary value problem below:

$$\begin{cases} u_{tt} + u_{xt} - 2u_{xx} = x^2 + t^2 & \text{in } (0,1) \times (0,\infty), \\ u(0,t) = a(t) & \text{on } \{x=0\} \times [0,\infty), \\ u(1,t) = b(t) & \text{on } \{x=1\} \times [0,\infty), \\ u(x,0) = \phi(x) & \text{on } [0,1] \times \{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } [0,1] \times \{t=0\}. \end{cases} \tag{61}$$

Show that a smooth solution to (61) must be unique.

Suggested Solutions:

(i) We start by factorizing the differential operator as follows:

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x \partial t} - \frac{\partial^2}{\partial x^2}\right) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x}\right).$$

Hence, the PDE reduces to

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x}\right) u = 0.$$

Consider the substitution of the form

$$\xi := x + t, \quad \eta := x - 2t.$$

One can then show by chain rule that

$$\begin{cases} \frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} - 2\frac{\partial}{\partial \eta} \end{cases}$$

and hence

$$\begin{cases} \frac{\partial}{\partial t} - \frac{\partial}{\partial x} &= -3\frac{\partial}{\partial \eta}, \\ \frac{\partial}{\partial t} + 2\frac{\partial}{\partial x} &= 3\frac{\partial}{\partial \xi}. \end{cases}$$

Hence, the PDE reduces to

$$-9u_{\xi\eta} = 0.$$

The general solution for this PDE is given by

$$u(\xi, \eta) = f(\xi) + g(\eta).$$

In the original variables, we have that the general solution is given by

$$u(x,t) = f(x+t) + g(x-2t).$$

To determine the functions $f$ and $g$, we apply the initial data as follows.

$$\begin{cases} \phi(x) = u(x,0) = f(x) + g(x), \\ \psi(x) = u_t(x,0) = f'(x) - 2g'(x). \end{cases}$$

Integrating the second equation from $0$, we have

$$\int_0^x \psi(s)\mathrm{d}s + f(0) - 2g(0) = f(x) - 2g(x).$$

Solving this together with the first equation, we have

$$\begin{cases} f(x) &= \frac{2}{3}\phi(x) + \frac{1}{3}\int_0^x \psi(s)\mathrm{d}s + \frac{1}{3}f(0) - \frac{2}{3}g(0), \\ g(x) &= \frac{1}{3}\phi(x) - \frac{1}{3}\int_0^x \psi(s)\mathrm{d}s - \frac{1}{3}f(0) + \frac{2}{3}g(0). \end{cases}$$

In other words, we have

$$\begin{cases} f(x+t) &= \frac{2}{3}\phi(x+t) + \frac{1}{3}\int_0^{x+t} \psi(s)\mathrm{d}s + \frac{1}{3}f(0) - \frac{2}{3}g(0), \\ g(x-2t) &= \frac{1}{3}\phi(x-2t) - \frac{1}{3}\int_0^{x-2t} \psi(s)\mathrm{d}s - \frac{1}{3}f(0) + \frac{2}{3}g(0). \end{cases}$$

Hence, we have

$$\begin{aligned} u(x,t) &= f(x+t) + g(x-2t) \\ &= \boxed{\frac{2}{3}\phi(x+t) + \frac{1}{3}\phi(x-2t) + \frac{1}{3}\int_{x-2t}^{x+t} \psi(s)\mathrm{d}s.} \end{aligned}$$

(ii) Let $u$ and $v$ be two different smooth solutions to (61). Let $w := u - v$. We have that $w$ satisfies the following PDE:

$$\begin{cases} w_{tt} + w_{xt} - 2w_{xx} = 0 & \text{in } (0,1)\times(0,\infty), \\ w(0,t) = 0 & \text{on } \{x=0\}\times[0,\infty), \\ w(1,t) = 0 & \text{on } \{x=1\}\times[0,\infty), \\ w(x,0) = 0 & \text{on } [0,1]\times\{t=0\}, \\ w_t(x,0) = 0 & \text{on } [0,1]\times\{t=0\}. \end{cases}$$

Consider the following energy:

$$E(t) = \frac{1}{2}\int_0^1 (w_t)^2(x,t) + 2(w_x)^2(x,t)\mathrm{d}x.$$

The coefficient of $w_x$ is chosen so that we have an exact cancellation in which we will see below. Hence, we compute

$$\begin{aligned} E'(t) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (w_t)^2(x,t) + 2(w_x)^2(x,t)\mathrm{d}x \\ &= \int_0^1 (w_t w_{tt} + 2w_x w_{xt})(x,t)\mathrm{d}x \\ &= \int_0^1 (w_t w_{tt} + 2w_x w_{tx})(x,t)\mathrm{d}x \quad \text{by Schwarz Theorem} \\ &= \int_0^1 (w_t w_{tt} - 2w_{xx} w_t)(x,t)\mathrm{d}x + 2w_x w_t|_{x=0}^{x=1} \quad \text{using Integrating by parts} \\ &= \int_0^1 w_t(w_{tt} - 2w_{xx})(x,t)\mathrm{d}x \quad \text{by boundary conditions} \\ &= -\int_0^1 (w_t w_{xt})(x,t)\mathrm{d}x \quad \text{by PDE} \\ &= -\int_0^1 (w_t w_{tx})(x,t)\mathrm{d}x \quad \text{by Schwarz} \\ &= -\int_0^1 \left(\frac{(w_t)^2}{2}\right)_x (x,t)\mathrm{d}x \quad \text{by reverse chain rule} \\ &= -\frac{(w_t)^2}{2}(1,t) + \frac{(w_t)^2}{2}(0,t) \quad \text{by Fundamental theorem of Calculus} \\ &= 0 \quad \text{by boundary conditions.} \end{aligned} \tag{62}$$

In the computations above, we have used the fact that $w_t(1,t) = w_t(0,t) = 0$. This follows from the fact that $w(0,t) = w(1,t) = 0$ and taking the derivative with respect to $t$.

With that, we have

$$E(t) = E(0) = 0 \quad \text{for all } t > 0.$$

since

$$E(0) = \frac{1}{2}\int_0^1 (w_t)^2(x,0) + 2(w_x)^2(x,0)\mathrm{d}x = 0$$

as $w_t(x,0) \equiv 0$ and $w_x(x,0) \equiv 0$ (which follows from $w(x,0)$ and taking partial derivative with respect to $x$). As $E(t) = 0$ for all $t > 0$ and the integrands are non-negative smooth functions[14], we must have that

$$w_t^2 = w_x^2 = 0 \quad \text{for all } (x,t) \in \mathbb{R} \times [0,\infty).$$

This implies

$$w_t = w_x = 0 \quad \text{for all } (x,t) \in \mathbb{R} \times [0,\infty).$$

Integrating $w_t = 0$ with respect to $t$ yields

$$w(x,t) = f(x) \quad \text{for some arbitrary function } f.$$

Plug this into $w_x = 0$, we have

$$f'(x) = 0$$

and hence $f(x) = C$ for some constant $C$. Hence, we have

$$w(x,t) = C \quad \text{for all } (x,t) \in \mathbb{R} \times [0,\infty).$$

In particular, since $w(x,0) = 0$, we must have the constant $C = 0$. Hence, we have

$$w \equiv 0 \text{ for all } (x,t) \in \mathbb{R} \times [0,\infty),$$

implying that $u \equiv v$, and hence we have the required uniqueness result.

Remark: Note that for a generic "wave-like" equation in which the differential operator is factorizable, one can convert it into a system of first order PDEs. For instance, if we set $v := u_t$ and $w := u_x$, then we have

$$w_t = u_{xt} = u_{tx} = v_x,$$

and

$$v_t = u_{tt} = -u_{xt} + 2u_{xx} = -v_x + 2w_x.$$

Hence, we have the following system of first order PDEs for $v$ and $w$ as follows:

$$\begin{pmatrix} w \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix}_x.$$

Such a formalism might be useful for numerical analysts as this means that a numerical scheme for vector-valued first-order PDEs would suffices to solve higher-order wave-like (hyperbolic) equations.

[14] In particular, if $w_x^2(\cdot,t)$ and $w_t^2(\cdot,t)$ are continuous in the first argument, it follows from a typical result in undergraduate analysis.

**Example 10.** (Spring 15, Problem 2, Hyperbolic, Modified.) Show that there is at most one solution to

$$\begin{cases} u_{tt}(x,t) - \partial_x\left(c(x)^2 u_x\right)(x,t) + u_t(x,t) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R}\times\{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{63}$$

where $\phi$ and $\psi$ are smooth and there exists some $\hat{c} > 0$ such that $|c(x)| < \hat{c}$.

Suggested Solutions: Let $u$ and $v$ be two solutions to (63) and define $w := u - v$. It follows that $w$ satisfies the following PDE:

$$\begin{cases} w_{tt}(x,t) - \partial_x\left(c(x)^2 w_x\right)(x,t) + w_t(x,t) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ w(x,0) = 0 & \text{on } \mathbb{R}\times\{t=0\}, \\ w_t(x,0) = 0 & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{64}$$

Consider the following energy on the whole space $\mathbb{R}$:

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (w_t)^2(x,t) + c(x)^2(w_x)^2(x,t)\mathrm{d}x. \tag{65}$$

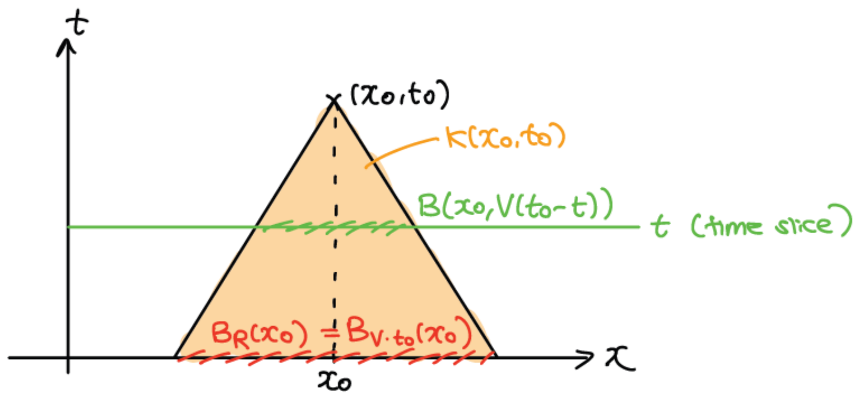
We compute its derivative as follows:

$$\begin{aligned} E'(t) &= \int_{-\infty}^{\infty} (w_t w_{tt})(x,t) + c(x)^2(w_x w_{xt})(x,t)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \left(w_t\partial_x\left(c(x)^2 w_x\right)\right)(x,t) - (w_t)^2(x,t) + c(x)^2(w_x w_{xt})(x,t)\mathrm{d}x \quad \text{by PDE} \\ &= \int_{-\infty}^{\infty} c(x)^2(w_{tx}w_x)(x,t) - (w_t)^2(x,t) + c(x)^2(w_x w_{xt})(x,t)\mathrm{d}x - c(\cdot)^2 w_t w_x(\cdot,t)\big|_{-\infty}^{\infty} \\ &= -\int_{-\infty}^{\infty} w_t^2(x,t)\mathrm{d}x \\ &\le 0, \end{aligned} \tag{66}$$

if $w$ has compact support (so that the boundary terms from applying integration by parts vanish). This would imply that $0 \le E(t) \le E(0) = 0$ and hence $w \equiv 0$ using a similar argument as in Example 9. Hence, it remains for us to prove a result on the domain of dependence as follows.

**Lemma.** (Finite Propagation Speed/Domain of Dependence.) Fix any $(x_0, t_0) \in \mathbb{R}\times(0,\infty)$. If $w \equiv w_t \equiv 0$ on $\{t=0\}\times B_{Vt_0}(x_0)$ for some $V > 0$,[15] then we have that $w \equiv w_t \equiv 0$ in the (backwards) cone $K(x_0,t_0)$ given by

$$K(x_0,t_0) = \{(x,t) \in \mathbb{R}\times[0,t_0] : |x - x_0| \le V(t_0 - t)\}.$$




[15] Here, $B_R(x_0) = \{x \in \mathbb{R} : |x - x_0| < R\}$ is the open ball of radius $R$ centered at $x_0$. This might also be represented as $B(x_0, R)$ too.

*Proof.* Consider the open ball obtained through a time slice of the cone $K(x_0, t_0)$ at some time $t \in [0, t_0]$. This is the open ball centered at $x_0$ given by $B(x_0, V(t_0 - t))$. We then consider the local energy given by

$$e(t) := \frac{1}{2}\int_{B(x_0,V(t_0-t))} w_t^2(x,t) + c(x)^2 w_x^2(x,t)\mathrm{d}x, \qquad \text{for } t \in [0, t_0]. \tag{67}$$

Taking the derivative and applying the Reynolds transport theorem, we have

$$e'(t) = \frac{1}{2}\int_{B(x_0,V(t_0-t))} \frac{\partial}{\partial t}\left(w_t^2(x,t) + c(x)^2 w_x^2(x,t)\right)\mathrm{d}x + \frac{1}{2}\int_{\partial B(x_0,V(t_0-t))} \left(w_t^2(x,t) + c(x)^2 w_x^2(x,t)\right)(\mathbf{v}\cdot\hat{\mathbf{n}})(x)\mathrm{d}S. \tag{68}$$

Here, $\mathbf{v}$ refers to the velocity vector corresponding to the moving domain $B(x_0, V(t_0 - t))$, while $\mathbf{n}$ is the outward-pointing unit normal to the domain. Since the ball is shrinking at the same speed $V$ at the ends of the ball, we have $\mathbf{v}\cdot\hat{\mathbf{n}} = -V$.
The first term is dealt as follows:

$$\begin{aligned}
&\frac{1}{2}\int_{B(x_0,V(t_0-t))} \frac{\partial}{\partial t}\left(w_t^2(x,t) + c(x)^2 w_x^2(x,t)\right)\mathrm{d}x \\
&= \int_{B(x_0,V(t_0-t))} \left(w_t w_{tt}(x,t) + c(x)^2 w_x w_{xt}(x,t)\right)\mathrm{d}x \\
&\underbrace{=}_{\text{PDE}} \int_{B(x_0,V(t_0-t))} \left(w_t \partial_x\left(c(x)^2 w_x\right)(x,t) - w_t^2(x,t) + c(x)^2 w_x w_{xt}(x,t)\right)\mathrm{d}x \\
&\leq \int_{B(x_0,V(t_0-t))} \left(w_t \partial_x\left(c(x)^2 w_x\right)(x,t) + c(x)^2 w_x w_{xt}(x,t)\right)\mathrm{d}x \\
&\underbrace{=}_{\text{IBP, Cancellations}} \int_{\partial B(x_0,V(t_0-t))} c(x)^2 (w_t w_x)(x,t)\mathrm{d}S. \\
&\underbrace{\leq}_{\text{Cauchy-Schwarz}} \int_{\partial B(x_0,V(t_0-t))} \frac{c(x)^2}{2}\left(w_t^2(x,t) + w_x^2(x,t)\right)\mathrm{d}S.
\end{aligned} \tag{69}$$

Note that Cauchy-Schwarz here refers to the inequality $|ab| \leq \frac{1}{2}(a^2 + b^2)$ for each $a, b \in \mathbb{R}$. Plugging this estimates into (68), we have

$$\begin{aligned}
e'(t) &\leq \frac{1}{2}\int_{\partial B(x_0,V(t_0-t))} c(x)^2\left(w_t^2(x,t) + w_x^2(x,t)\right) - V w_t^2(x,t) - V c(x)^2 w_x^2(x,t)\mathrm{d}S \\
&= \frac{1}{2}\int_{\partial B(x_0,V(t_0-t))} (c(x)^2 - V) w_t^2(x,t) + c(x)^2(1 - V) w_x^2(x,t)\mathrm{d}S \\
&\leq \frac{1}{2}\int_{\partial B(x_0,V(t_0-t))} (\hat{c}^2 - V) w_t^2(x,t) + c(x)^2(1 - V) w_x^2(x,t)\mathrm{d}S \\
&\leq 0
\end{aligned} \tag{70}$$

if we pick $V \geq \hat{c}^2$ and $V \geq 1$. Hence, pick $V = \max\{1, \hat{c}^2\}$. We have that $e'(t) \leq 0$ and hence

$$e(t) = e(0) = 0 \qquad \text{for all } t \in [0, t_0].$$

In particular, for all $(x, t) \in K(x_0, t_0)$, we have $w_x \equiv w \equiv w_t \equiv 0$. □

Remark: In the proof of the lemma above, the physical interpretation $V$ is the **estimated speed of propagation**. If we know that this is the minimum possible $V$, then this would be the actual **speed of propagation**. The proof is done using open balls rather than open intervals to allow for easier generalizability to wave equations on $\mathbb{R}^n$.

Note that if we have instead prove just the lemma, that by itself will allow us to deduce that $w \equiv 0$ for all $x \in \mathbb{R}, t > 0$.

In most qualifying problems (and possibly the exams for this class), you would be required to prove an analogous lemma on the finite speed of propagation of waves (this is in contrast with parabolic equations, which we will learn in a bit that it has an infinite speed of propagation!).

**Example 11.** Consider the wave equation:

$$\begin{cases} u_{tt}(x,t) = 16u_{xx}(x,t) & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = \begin{cases} x^2 - 255 & \text{for } |x| \le 16 \\ 0 & \text{otherwise} \end{cases} & \text{on } \mathbb{R}\times\{t=0\}, \\ u_t(x,0) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{otherwise} \end{cases} & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{71}$$

Compute

(i) $u(21,1)$,

(ii) $u(0,2)$, and

(iii) $u(8104,2022)$.

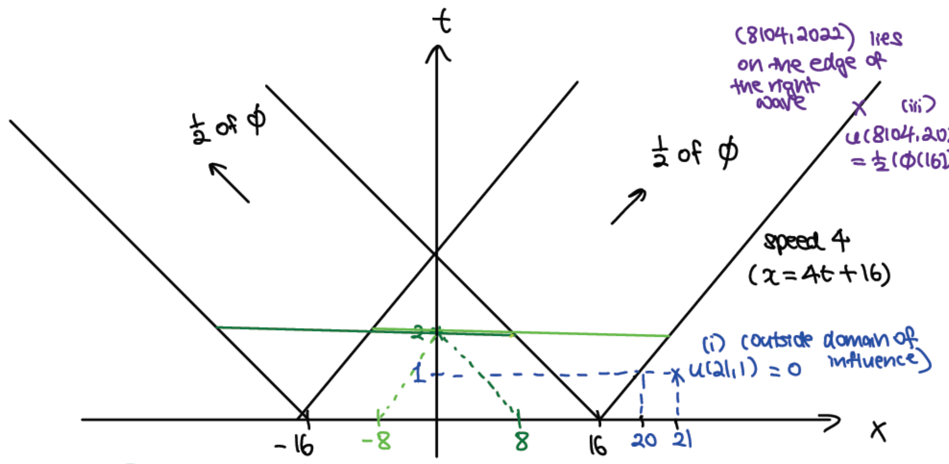
Hint: This is a question that is testing mainly the concept on domain of dependence/influence.

Suggested Solution: The wave speed is given by $c = \sqrt{16} = 4$. By d'Alembert's solution, explicitly given by

$$u(x,t) = \frac{1}{2}(\phi(x-4t) + \phi(x+4t)) + \frac{1}{8}\int_{x-4t}^{x+4t} \psi(y)\mathrm{d}y, \tag{72}$$

we can consider the contribution by $\phi$ and $\psi$ individually, and then add them up in the end!

$\underline{\phi.}$




(ii) ∴ $u(2,0) = \frac{1}{2}(\phi(8) + \phi(-8))$

$= \frac{1}{2}(8^2 - 255 + (-8)^2 - 255) = \underline{\underline{-191}}$

(iii) ie $8104 = 4(2022) + 16$.

∴ $u(8104,2022) = u(16,0) = \frac{1}{2}(\phi(16)) = \frac{1}{2}$.

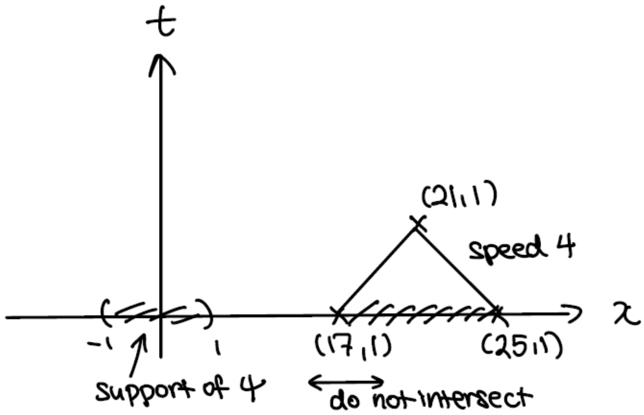
wave travelled at speed 4.

$\underline{\psi.}$ Note that the integrand $\psi$ is 1 if $|y| \le 1$. Thus, we can interpret the integral as $\frac{1}{8}\,|(x-4t, x+4t)\cap(-1,1)|$ (here $|\cdot|$ refers to the length of the interval). One can interpret this geometrically, or just notice that since we are given a bunch of points, we can just treat this as a bunch of computational exercise. Hence, we have

- At $(21,1)$, we have $\frac{1}{8}\,|(21-4, 21+4)\cap(-1,1)| = 0$.
- At $(0,2)$, we have $\frac{1}{8}\,|(-8,8)\cap(-1,1)| = \frac{2}{8} = \frac{1}{4}$.
- At $(8104,2022)$, we have

$$\frac{1}{8}\,|(8104 - 4\times 2022, 8104 + 4\times 2022)\cap(-1,1)| = \frac{1}{8}\,|(16,16192)\cap(-1,1)| = 0.$$

(Note that this is consistent with the concept of domain of dependence, which can be understood as follows. For a given $t$, and a point $x$, we look an interval given by $(x-4t, x+4t)$ in which $u$ depends on. This can be seen as a cone drawn backwards with speed $4$ and to see if this coincides with the support[16] of the initial data $\psi$ (which is $(-1,1)$). If they do not intersect, then $\psi$ did not impact this point $(x,t)$! See diagram below for a visualization of this, which helps you to visualize what you are computing!)




Summarizing, we then have

- $u(21,1) = 0 + 0 = 0$,
- $u(0,2) = -191 + \frac{1}{4} = -190.75$, and
- $u(8104, 2022) = \frac{1}{2} + 0 = \frac{1}{2}$.

[16] Support of a function refers to the set of points in the domain of the function in which the function does not vanish.

Qual problems for additional practice:

**Exercise 11.** (Fall 17, Problem 6, Hyperbolic.) Consider the wave equation

$$\begin{cases} u_{tt}(x,t) = c^2 \Delta u(x,t) & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = f(x) & \text{on } \mathbb{R}^3 \times \{t=0\}, \\ u_t(x,0) = g(x) & \text{on } \mathbb{R}^3 \times \{t=0\}. \end{cases}$$

The initial data $f$ and $g$ are only non-zero in the region $a < \|x\|_1 < b$, with $\|\cdot\|_1$ representing the $l^1$ norm of a vector in $\mathbb{R}^3$. Given a point $x$, find the time $T > 0$ such that $u(x,t) = 0$ for all $0 < t < T$ when

(i) $\|x\|_1 > b$,

(ii) $\|x\|_1 < a$.

**Exercise 12.** (Spring 17, Problem 3, Hyperbolic.) Let $u$ solve the following boundary value problem:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \{(t,x) \in (0,\infty) \times \mathbb{R}^3 : x_1 > t/2\}, \\ \qquad u_t = 4u_{x_1} & \text{on } \{x_1 = t/2\}. \end{cases}$$

Show that $u(x,t) = 0$ in $\{|x| < R - t\} \cap \{x_1 > t/2\}$ when $u(x,0) = u_t(x,0) = 0$ in $\{|x| < R\} \cap \{x_1 > 0\}$. Explain where the boundary condition on $\{x_1 = t/2\}$ has been used.

**Exercise 13.** (Spring 20, Problem 3, Hyperbolic.) Suppose that $u(x,t)$ is a piecewise $C^{2,2}$ solution to the forced wave equation:

$$u_{tt} = \Delta u + f(x,t) \quad \text{for } x \in \mathbb{R}^3, t > 0,$$

with initial conditions $u(x,0) = u_t(x,0) = 0$. Suppose moreover that the forcing function $f(x,t)$ is compactly supported on the spherical shell $1 \le \|x\| \le 2$.

(i) Find the compact support of $u$ at time $t > 0$. If you make use of any results about the region of influence for wave equation then you should prove them.

(ii) Now consider a spherically symmetric forcing function $f \equiv f(r)$ for $0 < t \le 1$, where $r = \|x\|$ and

$$f(r) = \begin{cases} \frac{1}{r} & \text{if } 1 < r < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Find an explicit formula for $u$ for $0 < t \le 1$.

Hint: You may find it useful to make use of the expression for the Laplacian of a spherically symmetric function:

$$\Delta u(r,t) = \frac{1}{r^2} \frac{\partial}{\partial r}\left( r^2 \frac{\partial u}{\partial r} \right).$$

**Exercise 14.** (Spring 21, Problem 4, Hyperbolic.) A field $u : [0, \infty)^2 \times [0, \infty) \to \mathbb{R}$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2},$$

with boundary conditions : $u = 0$ on $x_2 = 0$ and $\frac{\partial u}{\partial x_1} = 0$ on $x_1 = 0$.

Suppose that at $t = 0$, the functions $u(\cdot, 0)$ and $\frac{\partial u}{\partial t}(\cdot, 0)$ are compactly supported on $\sqrt{x_1^2 + x_2^2} < 1$. Find the compact support for $u$ at time $t$.

If you make use of results for the domain of dependence of a solution of the wave equation, then you should prove them.

**Exercise 15.** (Fall 21, Problem 4, Hyperbolic.) Show, for a constant $\beta \geq 0$, that the PDE:

$$u_{tt} + \beta u_t - u_{xx} + u = 0, \quad x \in \mathbb{R},$$

has at most one compactly supported solution if given $C^2$ initial data $u(x, 0) = \phi(x)$ and $u_t(x, 0) = \psi(x)$.

**Exercise 16.** (Spring 23, Problem 8, Hyperbolic.) Solve the initial value problem

$$u_{tt} - 2u_{xt} - 15u_{xx} = 0,$$

with $u(x, 0) = g(x)$ and $u_t(x, 0) = h(x)$.

Hint: Consider factoring the differential operator.

## 5 Discussion 5

I'll finish up the suggested solutions for the second example in the previous discussion supplement as follows.

**Example 12.** (Spring 15, Problem 2, Hyperbolic, Modified.) Show that there is at most one solution to

$$\begin{cases} u_{tt}(x,t) - \partial_x\left(c(x)^2 u_x\right)(x,t) + u_t(x,t) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R}\times\{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{73}$$

where $\phi$ and $\psi$ are smooth and there exists some $\hat{c} > 0$ such that $|c(x)| < \hat{c}$.

Suggested Solutions: Let $u$ and $v$ be two solutions to (73) and define $w := u - v$. It follows that $w$ satisfies the following PDE:

$$\begin{cases} w_{tt}(x,t) - \partial_x\left(c(x)^2 w_x\right)(x,t) + w_t(x,t) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ w(x,0) = 0 & \text{on } \mathbb{R}\times\{t=0\}, \\ w_t(x,0) = 0 & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{74}$$

Consider the following energy on the whole space $\mathbb{R}$:

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (w_t)^2(x,t) + c(x)^2 (w_x)^2(x,t)\mathrm{d}x. \tag{75}$$

We compute its derivative as follows:

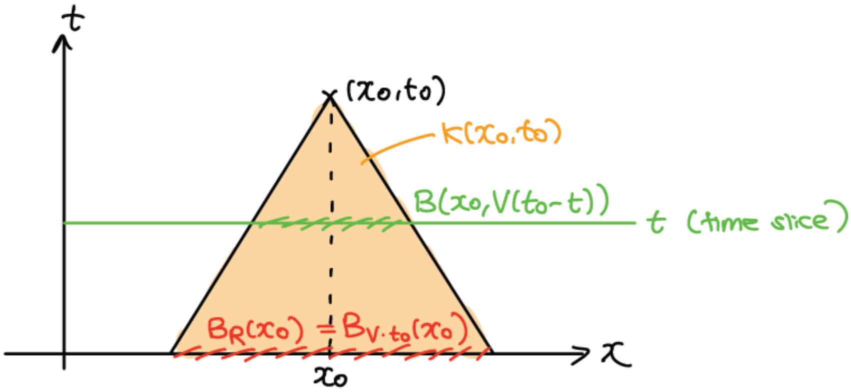
$$\begin{aligned} E'(t) &= \int_{-\infty}^{\infty} (w_t w_{tt})(x,t) + c(x)^2 (w_x w_{xt})(x,t)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \left(w_t \partial_x\left(c(x)^2 w_x\right)\right)(x,t) - (w_t)^2(x,t) + c(x)^2(w_x w_{xt})(x,t)\mathrm{d}x \qquad \text{by PDE} \\ &= \int_{-\infty}^{\infty} c(x)^2 (w_{tx} w_x)(x,t) - (w_t)^2(x,t) + c(x)^2 (w_x w_{xt})(x,t)\mathrm{d}x - c(\cdot)^2 w_t w_x(\cdot,t)\big|_{-\infty}^{\infty} \\ &= -\int_{-\infty}^{\infty} w_t^2(x,t)\mathrm{d}x \\ &\leq 0, \end{aligned} \tag{76}$$

if $w$ has compact support (so that the boundary terms from applying integration by parts vanish). This would imply that $0 \leq E(t) \leq E(0) = 0$ and hence $w \equiv 0$. Hence, it remains for us to prove a result on the domain of dependence as follows.

**Lemma.** (Finite Propagation Speed/Domain of Dependence.) Fix any $(x_0, t_0) \in \mathbb{R}\times(0,\infty)$. If $w \equiv w_t \equiv 0$ on $\{t = 0\} \times B_{Vt_0}(x_0)$ for some $V > 0$,[17] then we have that $w \equiv w_t \equiv 0$ in the (backwards) cone $K(x_0, t_0)$ given by

$$K(x_0, t_0) = \{(x,t) \in \mathbb{R}\times[0,t_0] : |x - x_0| \leq V(t_0 - t)\}.$$

[17] Here, $B_R(x_0) = \{x \in \mathbb{R} : |x - x_0| < R\}$ is the open ball of radius $R$ centered at $x_0$. This might also be represented as $B(x_0, R)$ too.




*Proof.* Consider the open ball obtained through a time slice of the cone $K(x_0,t_0)$ at some time $t \in [0,t_0]$. This is the open ball centered at $x_0$ given by $B(x_0, V(t_0-t))$. We then consider the local energy given by

$$e(t) := \frac{1}{2}\int_{B(x_0,V(t_0-t))} w_t^2(x,t) + c(x)^2 w_x^2(x,t)\mathrm{d}x, \qquad \text{for } t \in [0,t_0]. \tag{77}$$

Taking the derivative and applying the Reynolds transport theorem, we have

$$e'(t) = \frac{1}{2}\int_{B(x_0,V(t_0-t))} \frac{\partial}{\partial t}\left(w_t^2(x,t) + c(x)^2 w_x^2(x,t)\right)\mathrm{d}x + \frac{1}{2}\int_{\partial B(x_0,V(t_0-t))} \left(w_t^2(x,t) + c(x)^2 w_x^2(x,t)\right)(\mathbf{v}\cdot\hat{\mathbf{n}})(x)\mathrm{d}S. \tag{78}$$

Here, $\mathbf{v}$ refers to the velocity vector corresponding to the moving domain $B(x_0, V(t_0-t))$, while $\mathbf{n}$ is the outward-pointing unit normal to the domain. Since the ball is shrinking at the same speed $V$ at the ends of the ball, we have $\mathbf{v}\cdot\hat{\mathbf{n}} = -V$.

The first term is dealt as follows:

$$\begin{aligned}
&\frac{1}{2}\int_{B(x_0,V(t_0-t))} \frac{\partial}{\partial t}\left(w_t^2(x,t) + c(x)^2 w_x^2(x,t)\right)\mathrm{d}x \\
&= \int_{B(x_0,V(t_0-t))} \left(w_t w_{tt}(x,t) + c(x)^2 w_x w_{xt}(x,t)\right)\mathrm{d}x \\
&\underbrace{=}_{\text{PDE}} \int_{B(x_0,V(t_0-t))} \left(w_t \partial_x\left(c(x)^2 w_x\right)(x,t) - w_t^2(x,t) + c(x)^2 w_x w_{xt}(x,t)\right)\mathrm{d}x \\
&\leq \int_{B(x_0,V(t_0-t))} \left(w_t \partial_x\left(c(x)^2 w_x\right)(x,t) + c(x)^2 w_x w_{xt}(x,t)\right)\mathrm{d}x \\
&\underbrace{=}_{\text{IBP, Cancellations}} \int_{\partial B(x_0,V(t_0-t))} c(x)^2 (w_t w_x)(x,t)\mathrm{d}S. \\
&\underbrace{\leq}_{\text{Cauchy-Schwarz}} \int_{\partial B(x_0,V(t_0-t))} \frac{c(x)^2}{2}\left(w_t^2(x,t) + w_x^2(x,t)\right)\mathrm{d}S.
\end{aligned} \tag{79}$$

Note that Cauchy-Schwarz here refers to the inequality $|ab| \leq \frac{1}{2}(a^2+b^2)$ for each $a, b \in \mathbb{R}$. Plugging this estimates into (78), we have

$$\begin{aligned}
e'(t) &\leq \frac{1}{2}\int_{\partial B(x_0,V(t_0-t))} c(x)^2\left(w_t^2(x,t) + w_x^2(x,t)\right) - V w_t^2(x,t) - V c(x)^2 w_x^2(x,t)\mathrm{d}S \\
&= \frac{1}{2}\int_{\partial B(x_0,V(t_0-t))} (c(x)^2 - V) w_t^2(x,t) + c(x)^2(1-V) w_x^2(x,t)\mathrm{d}S \\
&\leq \frac{1}{2}\int_{\partial B(x_0,V(t_0-t))} (\hat{c}^2 - V) w_t^2(x,t) + c(x)^2(1-V) w_x^2(x,t)\mathrm{d}S \\
&\leq 0
\end{aligned} \tag{80}$$

if we pick $V \geq \hat{c}^2$ and $V \geq 1$. Hence, pick $V = \max\{1, \hat{c}^2\}$. We have that $e'(t) \leq 0$ and hence

$$e(t) = e(0) = 0 \qquad \text{for all } t \in [0,t_0].$$

In particular, for all $(x,t) \in K(x_0,t_0)$, we have $w_x \equiv w \equiv w_t \equiv 0$. □

Remark: In the proof of the lemma above, the physical interpretation $V$ is the **estimated speed of propagation**. If we know that this is the minimum possible $V$, then this would be the actual **speed of propagation**. The proof is done using open balls rather than open intervals to allow for easier generalizability to wave equations on $\mathbb{R}^n$.

Note that if we have instead prove just the lemma, that by itself will allow us to deduce that $w \equiv 0$ for all $x \in \mathbb{R}, t > 0$.

In most qualifying problems (and possibly the exams for this class), you would be required to prove an analogous lemma on the finite speed of propagation of waves (this is in contrast with parabolic equations, which we will learn in a bit that it has an infinite speed of propagation!).

Similarity Solutions.
(Shearer and Levy Chapter 5.1, Evans Chapter 2.3.1.)

In the derivation of the fundamental solution to the heat equation, one often searches for a self-similar solution of the form

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right), \quad x \in \mathbb{R}^n, t > 0.$$

This process is not limited to just the heat equation, but for any parabolic second-order PDE. The process to determine a self-similar solution to the PDE is as follows:

0. Consider a self similar solution of the form
$$u(x,t) = g(t) f\left(\frac{x}{h(t)}\right). \tag{81}$$
for some sufficiently regular functions $f, g$, and $h$.
1. Determine the functional form of $h(t)$ by dimensional analysis.
2. Use the auxiliary conditions (initial/boundary/others) to determine the functional form for $g(t)$.
3. Derive the corresponding ODE for $f$.
4. Determine the boundary conditions for the ODE for $f$.
5. Solve the ODE to obtain $f$.

The self-similar solution is thus given by (81).

The above process is better demonstrated with the use of an example, which we will observe below.

**Example 13.** (Spring 19, Problem 6, Parabolic.) A heated plate lies along the positive $x-$axis, and air flows over the plate in the $x-$direction. There is a simple shear flow above the plate. Assume that the plate surface is at a temperature $T_0$. Thus, for $x > 0$ and $y > 0$, the temperature field $T(x, y)$ above the plate is given by

$$\gamma y T_x(x, y) = D T_{yy}(x, y). \tag{82}$$

Here, $\gamma$ (the shear rate), and $D$ (diffusivity), and $T_0$ are constants. Assume that at $x = 0$ the air is at ambient temperature $T(0, y) = 0$, while on the surface of the plate, we have $T(x, 0) = T_0$ and far from the plate, we have $T(x, \infty) = 0$.

Construct a similarity solution to the PDE of the form:

$$T(x, y) = a(x) f\left(\frac{y}{L(x)}\right)$$

for $x > 0$ and $y > 0$. You should determine the functions $a(x), L(x)$, and $f(x)$.

Suggested Solutions:

Understanding the requirements of the question.

When we would like to find a similarity solution of the form $T(x, y) = a(x) f\left(\frac{y}{L(x)}\right)$, note that we are usually assuming that $\eta = \frac{y}{L(x)}$ is **unitless/dimensionless**, so $\eta$ represents a dimensionless variable. This also implies that $f\left(\eta = \frac{y}{L(x)}\right)$ is a function of a dimensionless variable, and thus dimensionless! Here, we denote $[X]$ to mean the "units of $X$".

One should note that although physically, $x$ and $y$ are spatial coordinates for this question and represents the same physical unit (of length), we shall assume for generalizability that they are "different".

Step 1: Determine the functional form of $L(x)$ by determining the units of $y$ in terms of $x$ and other physical constants.

This is because as mentioned, $\eta = \frac{y}{L(x)}$ is unitless, so $[y] = [L]$. Thus, to determine the units of $L$, it suffices to determine the units of $y$.

From the PDE, we have

$$\begin{aligned} [\gamma y T_x] &= [D T_{yy}] \\ \frac{[\gamma][y][T]}{[x]} &= \frac{[D][T]}{[y]^2} \\ [y] &= \left(\frac{[D][x]}{[\gamma]}\right)^{\frac{1}{3}}. \end{aligned} \tag{83}$$

This implies that $[L] = \left(\frac{[D][x]}{[\gamma]}\right)^{\frac{1}{3}}$, and thus, it makes physical sense to postulate

$$L(x) = \left(\frac{Dx}{\gamma}\right)^{\frac{1}{3}}. \tag{84}$$

Step 2: Use the auxiliary (initial/boundary/conservation of mass) condition to determine the functional form of $a(x)$.

For this question, we are given the boundary condition $T(x, 0) = T_0$. Substitute this into the form of our similiary solution, we have

$$T(x, 0) = a(x) f\left(\frac{0}{L(x)}\right) = a(x) f(0) = T_0 \tag{85}$$

Without loss of generality[18], we set $f(0) = 1$, and deduce that $a(x) = T_0$ for all $x \geq 0$.

[18] It really is. To see that it is the case, try setting $f(0) = 2$ and see if the final solution derived for the PDE changes.

Step 3: Derive the corresponding ODE for $f$.
Reaping our achievements in the previous two steps, we have

$$T(x,y) = T_0 f\left(\frac{y}{\left(\frac{Dx}{\gamma}\right)^{\frac{1}{3}}}\right) = T_0 f(\eta) \tag{86}$$

with $\eta(x,y) = \frac{y}{\left(\frac{Dx}{\gamma}\right)^{\frac{1}{3}}} = y\gamma^{\frac{1}{3}} D^{-\frac{1}{3}} x^{-\frac{1}{3}}$. Now, we have to compute $T_x$ and $T_{yy}$ using Chain rule, as described below:

- $\frac{\partial}{\partial x} T(x,y) = T_0 \frac{\partial}{\partial x} f(\eta(x,y)) = T_0 f'(\eta(x,y)) \frac{\partial \eta(x,y)}{\partial x} = -\frac{1}{3} T_0 f'(\eta(x,y)) y\gamma^{\frac{1}{3}} D^{-\frac{1}{3}} x^{-\frac{4}{3}}$.
- $\frac{\partial}{\partial y} T(x,y) = T_0 \frac{\partial}{\partial y} f(\eta(x,y)) = T_0 f'(\eta(x,y)) \frac{\partial \eta(x,y)}{\partial y} = T_0 f'(\eta(x,y)).\gamma^{\frac{1}{3}} D^{-\frac{1}{3}} x^{-\frac{1}{3}}$
- $$\begin{aligned} \frac{\partial^2}{\partial y^2} T(x,y) &= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} T(x,y)\right) = \frac{\partial}{\partial y}(T_0 f'(\eta(x,y))\gamma^{\frac{1}{3}} D^{-\frac{1}{3}} x^{-\frac{1}{3}}) \\ &= T_0 \gamma^{\frac{1}{3}} D^{-\frac{1}{3}} x^{-\frac{1}{3}} \frac{\partial}{\partial y}(f'(\eta(x,y))) = T_0 \gamma^{\frac{1}{3}} D^{-\frac{1}{3}} x^{-\frac{1}{3}} f''(\eta(x,y)) \frac{\partial \eta(x,y)}{\partial y} \\ &= T_0 \gamma^{\frac{2}{3}} D^{-\frac{2}{3}} x^{-\frac{2}{3}} f''(\eta(x,y)). \end{aligned}$$

Substitute these back to the PDE, we get (here, $f'$ represents $f'(\eta)$ and etc.)

$$\begin{aligned} \gamma y T_x &= D T_{yy} \\ \gamma y \left(-\frac{1}{3}\right) T_0 f' y \gamma^{\frac{1}{3}} D^{-\frac{1}{3}} x^{-\frac{4}{3}} &= D T_0 \gamma^{\frac{2}{3}} D^{\frac{2}{3}} x^{-\frac{2}{3}} f'' \\ -\frac{1}{3}\left(\frac{y^2 \gamma^{\frac{2}{3}}}{D^{-\frac{2}{3}} x^{\frac{2}{3}}}\right) f' &= f'' \\ -\frac{\eta^2}{3} f'(\eta) &= f''(\eta). \end{aligned} \tag{87}$$

Thus, we have obtained an ODE for $f$.

Remark: Remark: If your choice of $L(x)$ and $a(x)$ are correct, then the above must reduce to an ODE. If you can't cancel the physical constants such that you get just terms in $\eta$, you might have made a mistake in your choice of $L(x)$ and/or $a(x)$.

Step 4: Determine the corresponding boundary conditions for your ODE in $f$. The following shows the corresponding conversion of the given auxiliary boundary conditions to the corresponding boundary conditions for the ODE in $f$:

- $T(x,0) = T_0 \to T(x,0) = a(x) f(0) = T_0 f(0) = T_0$. This implies that $f(0) = 1$.
- $T(0,y) = T_0 f\left(\frac{y}{L(0)}\right) = T_0 f\left(\frac{y}{0}\right) = 0$. This implies that $\lim_{\eta\to\infty} f(\eta) = 0$ (ie "$f(\infty) = 0$").
- $T(x,\infty) = T_0 f\left(\frac{\infty}{L(x)}\right) = 0$. This also implies that $\lim_{\eta\to\infty} f(\eta) = 0$ (ie "$f(\infty) = 0$").

Step 5: Solve the given ODE: Let $g(\eta) = f'(\eta)$. This implies that we have

$$g'(\eta) = -\frac{\eta^2}{3} g(\eta). \tag{88}$$

One can solve this by separation of variables as follows:

$$\begin{aligned}\frac{\mathrm{d}g}{\mathrm{d}\eta} &= -\frac{\eta^2}{3}g \\ \frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}\eta} &= -\frac{\eta^2}{3} \\ \int \frac{1}{g}\mathrm{d}g &= \int -\frac{\eta^2}{3}\mathrm{d}\eta \\ \ln|g| &= -\frac{\eta^3}{9} + C \\ g(\eta) &= Ae^{-\frac{\eta^3}{9}}.\end{aligned} \tag{89}$$

Here, $A = \pm e^C$ is an arbitrary constant. Now, substitute back $f'(\eta) = g(\eta)$ to obtain

$$f'(\eta) = Ae^{-\frac{\eta^3}{9}}. \tag{90}$$

Recall that $f(0) = 1$, so we can use the Fundamental theorem of Calculus by integrating the above ODE with respect to $\eta$ starting from $0$. This yields

$$\begin{aligned}\int_0^\eta f'(\xi)\mathrm{d}\xi &= \int_0^\eta Ae^{-\frac{\xi^3}{9}}\mathrm{d}\xi \\ f(\eta) - f(0) &= A\int_0^\eta e^{-\frac{\xi^3}{9}}\mathrm{d}\xi \\ f(\eta) &= 1 + A\int_0^\eta e^{-\frac{\xi^3}{9}}\mathrm{d}\xi.\end{aligned} \tag{91}$$

Using the other boundary condition, ie $f(\infty) = 0$, we have

$$\begin{aligned}0 &= 1 + A\int_0^\infty e^{-\frac{\xi^3}{9}}\mathrm{d}\xi \\ A &= \frac{-1}{\int_0^\infty e^{-\frac{\xi^3}{9}}\mathrm{d}\xi}.\end{aligned} \tag{92}$$

This implies that we have

$$\begin{aligned}f(\eta) &= 1 - \frac{\int_0^\eta e^{-\frac{\xi^3}{9}}\mathrm{d}\xi}{\int_0^\infty e^{-\frac{\xi^3}{9}}\mathrm{d}\xi} \\ f\left(\frac{y}{\left(\frac{Dx}{\gamma}\right)^{\frac{1}{3}}}\right) &= 1 - \frac{\int_0^{\left(\frac{y}{\left(\frac{Dx}{\gamma}\right)^{\frac{1}{3}}}\right)} e^{-\frac{\xi^3}{9}}\mathrm{d}\xi}{\int_0^\infty e^{-\frac{\xi^3}{9}}\mathrm{d}\xi}.\end{aligned} \tag{93}$$

The final similarity solution for the PDE is thus given by

$$T(x,y) = T_0 f(\eta(x,y)) \tag{94}$$

and hence

$$T(x,y) = \boxed{T_0\left(1 - \frac{\int_0^{\left(\frac{y}{\left(\frac{Dx}{\gamma}\right)^{\frac{1}{3}}}\right)} e^{-\frac{\xi^3}{9}}\mathrm{d}\xi}{\int_0^\infty e^{-\frac{\xi^3}{9}}\mathrm{d}\xi}\right)}.$$

Qual problems for additional practice:

**Exercise 17.** (Fall 17, Problem 2, Parabolic.) A chemical diffuses freely in 1D, satisfying the following PDE:

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + \delta(x)\Theta(t),$$

where $\delta(x)$ refers to the delta-function with a point source at $x = 0$, and $\Theta(t)$ is the Heaviside function with $\Theta(t) = 1$ for $t > 0$ and $0$ otherwise. You may assume that $c(x, 0) = 0$ for all $x \neq 0$, and that $\lim_{x\to\pm\infty} c(x, t) = 0$.

**Exercise 18.** (Spring 17, Problem 8, Parabolic.) The space $y > 0$ is filled with a non-Newtonian fluid, initially at rest. A plate at $y = 0$ is set into motion at time $t = 0$. The fluid velocity, $u(t, y)$ then obeys an equation:

$$\frac{\partial u}{\partial t} = -\frac{\partial \tau}{\partial y}, \quad t > 0, y > 0$$

with boundary conditions:

$$u(t, 0) = 1, \quad \text{and} \quad u(t, +\infty) = 0$$

and initial condition $u(0, y) = 0$ for $y > 0$. Here, $\tau$ is assumed to obeys a constitutive equation:

$$\tau = \left(\frac{\partial u}{\partial y}\right)^2$$

(i) Try to derive a similarity solution i.e. look for a solution of the form $u(t, y) = f(\eta)$ where $\eta = y/\delta(t)$ for some function $\delta(t)$, that you will need to determine), by applying only the boundary condition $u(t, 0) = 1$. Show that this similarity solution can not be compatible with the other boundary condition, or with the initial condition.

(ii) To find a solution that is compatible with all boundary and initial conditions we modify the constitutive equation to:

$$\tau = \begin{cases} \left(\frac{\partial u}{\partial y}\right)^2 & \text{if } \frac{\partial u}{\partial y} < 0 \\ 0 & \text{if } \frac{\partial u}{\partial y} \geq 0 \end{cases}$$

Derive a similarity solution that satisfies all of the initial and boundary conditions.

Hint: Start by assuming that the solution breaks down into two parts: $0 < y < Y(t)$, in which $\tau \neq 0$ and $y > Y(t)$ in which $\tau = 0$. Derive continuity conditions that must be applied at $y = Y(t)$. You need to solve for the function $Y(t)$, as well as for $f(\eta)$.

**Exercise 19.** (Spring 21, Problem 3, Parabolic.) The height $h(r,t) \geq 0$ of a spreading axisymmetric droplet obeys the following equation:

$$\frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial h}{\partial r}\right), 0 < r < L(t),$$

along with the boundary condition $h(L(t),t) = 0$ and volume constraint $\int_0^{L(t)} rhdr = 1$. Here $r$ is the distance from the center of the drop and $\{r \leq L(t)\}$ equals the support of the drop at time $t$.

Assume that $h$ is a similarity solution of the form $h(r,t) = t^\alpha H\left(r/t^\beta\right)$ and $L(t) = \eta_0 t^\beta$, and solve the differential equation to find the function $H$ and the constants $\alpha$ and $\beta$ explicitly. We assume that $h$ is smooth in its support.

You should present an equation solved by $\eta_0$, though it is not necessary to solve for $\eta_0$ explicitly.

**Exercise 20.** (Spring 22, Problem 5, Parabolic.) Suppose that $u(x,t)$ satisfies the porous-medium PDE on an expanding domain given by

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^2\frac{\partial u}{\partial x}\right), \quad |x| < L(t), t > 0$$

with boundary conditions $u(\pm L(t), 0) = 0$ and total-mass condition $\int_{-L(t)}^{L(t)} u(x,t)\mathrm{d}x = 1$.

Find a similarity solution of the form $u(x,t) = t^a f\left(\frac{x}{t^b}\right)$ and $L(t) = ct^b$. Your solution should include the values of $a, b,$ and $c$ and an explicit expression for the function $f$.

# 6 Discussion 6

Fundamental Solution to the Heat Equation.
(Evans Chapter 2.3, Shearer and Levy Chapter 5.)
‘ Consider the heat equation in $\mathbb{R}^n$ below:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases} \tag{95}$$

The solution is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) \mathrm{d}y, \tag{96}$$

with $\Phi(x,t)$ as the **fundamental solution** to the heat equation and its explicit expression is given by

$$\Phi(x,t) := \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) \quad \text{for } t > 0. \tag{97}$$

Some properties/notes:

- We have
$$\int_{\mathbb{R}^n} \Phi(x,t) \mathrm{d}x = 1 \quad \text{for all } t > 0.$$
- We can view $\Phi(x,t)$ as the solution to the heat equation with the Dirac delta function (or in mathematical term, the Dirac measure) centered at $x=0$ as the initial data. In fact, since $\int_0^\infty u(x,t)\mathrm{d}t$ is constant in time, we do expect the first property that we've mentioned. Furthermore, by viewing (97) as the convolution of the fundamental solution with respect to the initial data, this returns the fundamental solution as the solution to the heat equation with the Dirac delta function as the initial data as it is the identity for convolutions.

On the other hand, for the **inhomogeneous heat equation** in $\mathbb{R}^n$ given by

$$\begin{cases} u_t - \Delta u = g(x,t) & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = 0 & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases} \tag{98}$$

**Duhamel's principle** gives the solution to (97), with its explicit expression given by

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) \mathrm{d}y \mathrm{d}s. \tag{99}$$

Hence, for the inhomogeneous heat equation with generic initial data given by

$$\begin{cases} u_t - \Delta u = f(x,t) & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases} \tag{100}$$

by linearity, we have

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) \mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) \mathrm{d}y \mathrm{d}s. \tag{101}$$

In addition to the explicit formulas for the solution to the heat equations above, more often than not, the qualifying exam problems would require the application of $\geq 1$ of the following:

- Substitution with a decay factor; i.e by viewing
$$u_t + u + ... = 0 \implies (e^t u)_t + e^t(...) = 0$$
- Substitution representing a change of reference;
$$u_t + au_x + ...(x,t) = 0 \quad \underbrace{\implies}_{\text{Set } x'=x-at, t'=t} \quad u_{t'} + ...(x',t') = 0$$
- Appropriate substitution to move initial data/boundary terms to the PDE (or vice versa).
- Even/odd extension. This is for solving initial-boundary value problems (or what Shearer and Levy label as the quarter-plane problem), with conditions on the solution imposed on the boundary. (Usually for 1D problems.)

We will see examples of three of the above all in action in a single qualifying exam problem below.

**Example 14.** (Fall 18, Problem 5, Parabolic.) Consider the following initial-boundary value problem for $u = u(x,t)$ in the domain $\{x > 0\} \times \{t > 0\}$ :

$$\begin{cases} u_t - u_{xx} + au = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x) & \text{on } \mathbb{R}^+ \times \{t = 0\}, \\ u(0,t) = g(t) & \text{on } \{x = 0\} \times \{t > 0\}, \end{cases} \tag{102}$$

where $f(x)$ and $g(t)$ are continuous functions with compact support and $a$ is a constant. Find an explicit solution to this problem.

Suggested Solutions: First, observe that $u_t(x,t) + au(x,t)$ can be made into the form $\left(u(x,t)e^{at}\right)_t$ by multiplying the factor $e^{at}$ throughout. Hence (together with the fact that $(ue^{at})_{xx} = e^{at}u_{xx}$), we have

$$\begin{cases} \left(ue^{at}\right)_t - \left(ue^{at}\right)_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x) & \text{on } \mathbb{R}^+ \times \{t = 0\}, \\ u(0,t) = g(t) & \text{on } \{x = 0\} \times \{t > 0\}. \end{cases} \tag{103}$$

This implies that we would have to transform the initial data and boundary conditions accordingly. Here, we do as follows:

$$\begin{cases} \left(ue^{at}\right)_t - \left(ue^{at}\right)_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ (ue^{at})(x,0) = f(x) & \text{on } \mathbb{R}^+ \times \{t = 0\}, \\ (ue^{at})(0,t) = g(t)e^{at} & \text{on } \{x = 0\} \times \{t > 0\}. \end{cases} \tag{104}$$

Now, set $v(x,t) = u(x,t)e^{at}$. The above PDE becomes

$$\begin{cases} v_t - v_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ v(x,0) = f(x) & \text{on } \mathbb{R}^+ \times \{t = 0\}, \\ v(0,t) = g(t)e^{at} & \text{on } \{x = 0\} \times \{t > 0\}. \end{cases} \tag{105}$$

To deal with the fact that the boundary term along $x = 0$ is non-zero, we move the term $g(t)e^{at}$ to the left to obtain

$$\begin{cases} v_t - v_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ v(x,0) = f(x) & \text{on } \mathbb{R}^+ \times \{t = 0\}, \\ v(0,t) - g(t)e^{at} = 0 & \text{on } \{x = 0\} \times \{t > 0\}. \end{cases} \tag{106}$$

We can perform the corresponding transformation for the initial data and the PDE as follows:

$$\begin{cases} (v - g(t)e^{at})_t + (g(t)e^{at})_t - (v - g(t)e^{at})_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ v(x,0) - g(0) = f(x) - g(0) & \text{on } \mathbb{R}^+ \times \{t = 0\}, \\ v(0,t) - g(t)e^{at} = 0 & \text{on } \{x = 0\} \times \{t > 0\}. \end{cases} \tag{107}$$

Hence, if we perform the substitution $w(x,t) = v(x,t) - g(t)e^{at}$, the above PDE reduces to

$$\begin{cases} w_t - w_{xx} = -(g(t)e^{at})_t & \text{in } \mathbb{R} \times (0,\infty), \\ w(x,0) = f(x) - g(0) & \text{on } \mathbb{R}^+ \times \{t = 0\}, \\ w(0,t) = 0 & \text{on } \{x = 0\} \times \{t > 0\}. \end{cases} \tag{108}$$

With the Dirichlet boundary condition along $x = 0$, we perform an odd extension of the initial data as follows. Let $h(x) = f(x) - g(0)$ and $\hat{h}(x)$ denote the odd extension of $h(x)$. The above PDE then reduces to solving

$$\begin{cases} w_t - w_{xx} = -(g(t)e^{at})_t & \text{in } \mathbb{R} \times (0,\infty), \\ w(x,0) = \hat{h}(x). & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \tag{109}$$

By the Duhamel's principle for the inhomogeneous heat equation, we then have

$$w(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)\hat{h}(y)\mathrm{d}y - \int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s) \cdot (g(s)e^{as})'\mathrm{d}y\mathrm{d}s. \tag{110}$$

Undoing all the transformations above, we have

$$u(x,t) = e^{-at}v(x,t) = e^{-at}\left(w(x.t) - g(t)e^{at}\right) = e^{-at}w(x,t) - g(t)$$

and hence

$$u(x,t) = \boxed{e^{-at}\int_{\mathbb{R}} \Phi(x-y,t)\hat{h}(y)\mathrm{d}y - e^{-at}\int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s)\cdot(g(s)e^{as})'\mathrm{d}y\mathrm{d}s - g(t)}. \tag{111}$$

with $h(x) = f(x) - g(0)$ and $\hat{h}$ representing the odd extension of $h$.

Midterm Review.

First-order non-linear PDEs:

- Method of Characteristics.
- When does it fail and the earliest time for shock formation (discontinuous solutions).

Hyperbolic PDEs:

- Factorizing operators into linear factors and deriving general solutions.
- Domain of dependence and influence.
- Energy method for uniqueness.
- Energy method for domain of dependence.

**Example 15.** (Spring 15, Problem 3, Hyperbolic.) Solve for $\mathbf{u} : [0,\infty)\times\mathbb{R}^2$ that satisfies

$$\frac{\partial u_i}{\partial t}(t,\mathbf{x}) + \sum_{j=1}^{2}\frac{\partial u_i}{\partial x_j}(t,\mathbf{x})u_j(t,\mathbf{x}) = -u_i(t,\mathbf{x}) \quad \mathbf{x}\in\mathbb{R}^2, t\in(0,\infty), \text{ for each } i$$

with initial conditions

$$\mathbf{u}(0,\mathbf{x}) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \text{ where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Suggested Solutions: Writing out the PDEs explicitly, we have

$$\begin{aligned}\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} &= -u_1,\\ \frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} &= -u_2.\end{aligned}$$

Hence, observe that if we parameterize the coordinates $x_1(t)$ and $x_2(t)$ using just one parameter - time, we have

$$\begin{aligned}\frac{\mathrm{d}u_1}{\mathrm{d}t}(t,x_1(t),x_2(t)) &= \frac{\partial u_1}{\partial t} + \frac{\mathrm{d}x_1(t)}{\mathrm{d}t}\frac{\partial u_1}{\partial x_1} + \frac{\mathrm{d}x_2(t)}{\mathrm{d}t}\frac{\partial u_1}{\partial x_2},\\ \frac{\mathrm{d}u_2}{\mathrm{d}t}(t,x_1(t),x_2(t)) &= \frac{\partial u_2}{\partial t} + \frac{\mathrm{d}x_1(t)}{\mathrm{d}t}\frac{\partial u_2}{\partial x_1} + \frac{\mathrm{d}x_2(t)}{\mathrm{d}t}\frac{\partial u_2}{\partial x_2}.\end{aligned}$$

Hence, both $u_1$ and $u_2$ share the same characteristics, defined by the ODEs:

$$\begin{aligned}\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= u_1(t),\\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= u_2(t),\end{aligned} \tag{112}$$

while $u_1(t) := u_1(t,x_1(t),x_2(t))$ and $u_2(t) := u_2(t,x_1(t),x_2(t))$ by the PDEs satisfy

$$\begin{aligned}\frac{\mathrm{d}u_1(t)}{\mathrm{d}t} &= -u_1(t),\\ \frac{\mathrm{d}u_2(t)}{\mathrm{d}t} &= -u_2(t).\end{aligned} \tag{113}$$

By solving the ODEs in (113), we have

$$\begin{aligned}u_1(t) &= u_1(0)e^{-t},\\ u_2(t) &= u_2(0)e^{-t}.\end{aligned} \tag{114}$$

Plugging these into (112), we have

$$\begin{aligned}\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= u_1(0)e^{-t},\\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= u_2(0)e^{-t},\end{aligned} \tag{115}$$

which we can solve to obtain

$$\begin{aligned}x_1(t) &= x_1(0) + u_1(0)(1-e^{-t}),\\ x_2(t) &= x_2(0) + u_2(0)(1-e^{-t}).\end{aligned} \tag{116}$$

By utilizing the initial data, we have

$$\begin{aligned}u_1(0) &= u_1(0,x_1(0),x_2(0)) = -x_2(0),\\ u_2(0) &= u_2(0,x_1(0),x_2(0)) = x_1(0).\end{aligned} \tag{117}$$

Substituting these into (116), we have

$$\begin{aligned}x_1(t) &= x_1(0) - x_2(0)(1-e^{-t}),\\ x_2(t) &= x_2(0) + x_1(0)(1-e^{-t}).\end{aligned} \tag{118}$$

We can rewrite this as the following matrix equation

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & -(1-e^{-t}) \\ 1-e^{-t} & 1 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}.$$

Upon inversion, we obtain

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \frac{1}{1+(1-e^{-t})^2} \begin{pmatrix} 1 & 1-e^{-t} \\ -(1-e^{-t}) & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}. \tag{119}$$

Hence, by the method of characteristics, we have

$$\begin{aligned} \mathbf{u}(t,\mathbf{x}) \underbrace{:=}_{\text{along char}} \mathbf{u}(t,\mathbf{x}(t)) \underbrace{=}_{(114)} e^{-t}\mathbf{u}(0,\mathbf{x}(0)) \underbrace{=}_{(117)} e^{-t} \begin{pmatrix} -x_2(0) \\ x_1(0) \end{pmatrix} = e^{-t} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} \\ \underbrace{=}_{(119)} \frac{e^{-t}}{1+(1-e^{-t})^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1-e^{-t} \\ -(1-e^{-t}) & 1 \end{pmatrix} \mathbf{x}. \end{aligned} \tag{120}$$

**Example 16.** (Fall 21, Problem 3, Hyperbolic.) Show, for a constant $\beta \geq 0$, that the PDE:

$$u_{tt} + \beta u_t - u_{xx} + u = 0, \quad x \in \mathbb{R},$$

has at most one compactly supported solution if given $C^2$ initial data $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$.

Suggested Solutions:

Let $u$ and $v$ both be compactly supported (smooth) solutions to the PDE. Define $w := u - v$. Then, $w$ has compact support (with $\operatorname{supp}(w) \subset \operatorname{supp}(u) \cup \operatorname{supp}(v)$) and solves the PDE:

$$\begin{cases} w_{tt} + \beta w_t - w_{xx} + w = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ w(x,0) = 0 & \text{on } \mathbb{R} \times \{t = 0\}, \\ w_t(x,0) = 0 & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

Consider the following energy:

$$E(t) = \frac{1}{2} \int_{\mathbb{R}} (w_t)^2(x,t) + (w_x)^2(x,t) + w^2(x,t) \mathrm{d}x.$$

We compute the time derivative of the energy as follows:

$$\begin{aligned} E'(t) &= \int_{\mathbb{R}} w_t w_{tt} + w_x w_{xt} + w w_t \mathrm{d}x \\ &= \int_{\mathbb{R}} w_t \left(-\beta w_t + w_{xx} - w\right) + w_x w_{xt} + w w_t \mathrm{d}x \\ &= -\beta \int_{\mathbb{R}} (w_t)^2(x,t) \mathrm{d}x + \int_{\mathbb{R}} w_t w_{xx}(x,t) + w_x w_{xt}(x,t) \mathrm{d}x \\ &\underbrace{=}_{\text{IBP}} -\beta \int_{\mathbb{R}} (w_t)^2(x,t) \mathrm{d}x + w_t w_x(\cdot,t)\big|_{-\infty}^{\infty} + \int_{\mathbb{R}} -w_{xt} w_x(x,t) + w_x w_{xt}(x,t) \mathrm{d}x \\ &= -\beta \int_{\mathbb{R}} (w_t)^2(x,t) \mathrm{d}x + w_t w_x(\cdot,t)\big|_{-\infty}^{\infty} \\ &\leq 0. \end{aligned}$$

The last inequality follows from the fact that $\beta \geq 0$ and the boundary term vanishes as $w$ is assumed to have compact support.

Hence, as $0 \leq E(t) \leq E(0) = 0$, we conclude that $E(t) = 0$ for all $t > 0$ and thus $w \equiv 0$. Hence, $u \equiv v$, implying that the PDE has at most one compactly supported solution.

Remark: Note that the above argument also works for $\beta < 0$, but this requires a slight modification to the argument above. Can you see how to implement the aforementioned modification?

Qual problems for additional practice:

**Exercise 21.** (Fall 19, Problem 6, Parabolic.)

Consider the heat equation $u_t = u_{xx}$, with $x \in \mathbb{R}$ and initial condition $u(x,0) = H(x)$, where $H$ denotes the Heaviside function (which is equal to 0 for $x < 0$ and to 1 for $x > 0$ ).

(i) Prove that the solution to this problem has infinite speed of propagation. That is, show for any $y < 0$ and any time $t > 0$, that $u(y,t) > 0$.

*[Hint: Consider the formula for the solution using the heat kernel.]*

(ii) Consider the same initial condition, but now with the evolution equation

$$u_t = -u_{xxxx}.$$

Prove that the solution to this problem also has infinite speed of propagation.

*[Hint: Consider the kernel $k_4(x,t)$ whose Fourier transform satisfies*

$$\widehat{k_4(x,t)} = \exp\left(-16\pi^4\xi^4 t\right)$$

*You may assume the following properties of $k_4$ : it decays at infinity, it is symmetric, and it is self-similar (specifically, $k_4(x,t) = K_4\left(x/t^{1/4}\right)$ ). You may use without proof the fact that $K_4$ changes sign and that it decays faster than any polynomial. ]*

**Exercise 22.** (Spring 20, Problem 5, Parabolic.)

Consider the heat equation

$$u_t = \Delta u, \quad u_{t=0} = u_0$$

with initial data $0 \leq u_0 \leq M$, and $u_0 \in \mathcal{C}_0^\infty\left(\mathbb{R}^n\right)$. (Recall that the notation $\mathcal{C}_0^\infty$ means that a function is smooth and decays to 0 at $\infty$ ).

(i) Prove that for any $T > 0$, the solution $u(x,t)$ satisfies $0 \leq u(x,t) \leq M$. Hint - use the explicit form of the solution of the heat equation.

(ii) Consider a nonnegative function $\phi \in \mathcal{C}_0^\infty\left(\mathbb{R}^n\right)$. Define $\phi_\epsilon = \frac{1}{\epsilon^n}\phi(x/\epsilon)$ with $\int_{\mathbb{R}^n}\phi \mathrm{d}x = 1$, and define $u_\epsilon(x) = \phi_\epsilon * u_0$ with $u_0$ defined as above. Prove that $u_\epsilon(x)$ satisfies $0 \leq u_\epsilon(x) \leq M$.

**Exercise 23.** (Spring 22, Problem 4, Parabolic.)

Let $u(x,t;y)$, with $x,y,t > 0$, be a Green's function solution of the partial differential equation

$$\frac{\partial u}{\partial t} = u + \frac{\partial^2 u}{\partial x^2},$$

with boundary conditions $u(0,t;y) = 0, u(\infty,t;y) = 0$ and initial condition $u(x,0;y) = \delta(x-y)$. By explicitly deriving a formula for the solution $u$, show that it satisfies the reciprocity property $u(x,t;y) = u(y,t;x)$.

*[Note: If you make use of the fundamental solution of the heat equation in your solution, then you should state its formula. However, you do not need to prove it.]*

# 7 Discussion 7

Maximum principles for (Nonlinear) Elliptic and Parabolic PDEs.

Maximum principles in general are results stating that the maximum of a solution must be attained at its boundary (or a parabolic boundary for a parabolic equation). These can be used to prove some estimates on the solutions in the interior domain, obtain uniqueness, and possibly continuous dependence on initial data.

Essentially, maximum principles rely on the fact that a local maximum cannot be attained in the interior region. If so, one would observe a contradiction in "sign", and hence is essentially a sign argument.
For example, at an interior maximum, $u_t \geq 0$ ($= 0$ for elliptic equations and parabolic equations not at $t = T$ slice, $\geq 0$ along the $t = T$ slice), $\Delta u \leq 0$, and observe a contradiction using the PDE. However, for most instances, there is a possibility of "degenerate solutions" and thus solutions agreeing in sign on both sides of the PDEs. For example:

- Heat Equation: $\underbrace{u_t}_{\geq 0} = \underbrace{\Delta u}_{\leq 0}$.
- Laplace Equation: $-\underbrace{\Delta u}_{\leq 0} = 0$.

In most cases, the method that we can use to resolve this is by means of a small perturbation that creates a small positive term on the left (or a certain non-zero term that supports the sign of the PDE on either sides). We will see examples of this in Example 18 and 19. Possible perturbations include (for some $C \in \mathbb{R}$ independent of $\varepsilon$):

- $\pm\varepsilon e^{\pm Ct}$,
- $\pm\varepsilon t$,
- $\pm\varepsilon e^{\pm Cx_i}$,
- $\pm\varepsilon e^{\pm C\|x\|^2}$,
- $\pm\varepsilon\|x\|^2$, ...

As mentioned above, the correct choice of sign requires considering the correct perturbation such that we can create an actual contradiction.

Note that in some cases, one might have to use an argument called the "first contact argument". An example of this is in Example 17.

**Example 17.** (Spring 20, Problem 4, Parabolic) Let $u(x,t)$ a $C^{2,1}$ solution to the equation

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u^2(x,t) - u(x,t) & \text{in } U \times (0,\infty), \\ u(\cdot,t)|_{\partial U} = 0 & \text{on } \partial U \times [0,\infty), \\ u(x,0) = g(x) & \text{on } U \times \{t=0\}, \end{cases} \tag{121}$$

where $U \subset \mathbb{R}^d$ is a bounded open domain, and its boundary $\partial U$ is piecewise smooth. Suppose that $0 \le g(x) \le \kappa < 1$ for each $x \in U$. Prove that for all $(x,t) \in U \times (0,\infty)$

(i) $0 \le u(x,t) \le \kappa$, and

(ii) $\int_U u(x,t)^2 \mathrm{d}x \le \left(\int_U g(x)^2 \mathrm{d}x\right) e^{2(\kappa-1)t}$.

Remark: Recall that we have done (ii) in Example 3 of Supplement 3. Hence, it suffices to show (i).

Suggested Solutions for (i):

$0 \le u(x,t)$ for all $(x,t) \in U \times (0,\infty)$. (Note that it makes sense to not include $t=0$ since we already know that $0 \le u(x,0) = g(x) \le \kappa < 1$, so there is no need to prove this.)

Suppose for a contradiction that there exists some $(x_1,t_1) \in U \times (0,\infty)$ such that $u(x_1,t_1) < 0$.
Pick the "cylindrical" spacetime domain given by

$$\Omega_T := U \times (0,T) \tag{122}$$

for some $T > t_1$. By the compactness of $\overline{\Omega_T} = \overline{U} \times [0,T]$, there exists $(x_2,t_2) \in \overline{\Omega_T}$ such that the minimum is attained. Since $(x_1,t_1) \in \overline{\Omega_T}$ and $(x_2,t_2)$ attains the minimum, we have

$$u(x_2,t_2) \le u(x_1,t_1) < 0. \tag{123}$$

However, since $u(\cdot,t)|_{\partial U} = 0$ for all $t > 0$, by (123), $(x_2,t_2)$ cannot be on $\partial U$. Similarly, since $u(x,0) = g(x) \ge 0$, $t_2 \ne 0$ too. This implies that $(x_2,t_2) \in \Omega_T$ instead.

As a minimizing point $(x_2,t_2)$ obtained in the interior of a compact set or along $t_2 = T$, we must have

- $u_t(x_2,t_2) \le 0$ ($= 0$ if it is in the interior, or $\le 0$ along $t = t_2 = T$ since as a minimum point, we must have $u(x_2,t_2) \le u(x_2,s_2)$ for any $s_2 < t_2$), and
- $\Delta u(x_2,t_2) \ge 0$, with
- $u^2(x_2,t_2) - u(x_2,t_2) > 0$ since $u(x_2,t_2) < 0$ (ie $y^2 - y > 0$ for all $y < 0$).

Plugging these into the PDE at $(x_2,t_2)$, we have

$$\underbrace{u_t(x,t)}_{\le 0} = \underbrace{\Delta u(x,t)}_{\ge 0} + \underbrace{u^2(x,t) - u(x,t)}_{>0}, \tag{124}$$

a contradiction.

$u(x,t) \le \kappa$ for all $(x,t) \in U \times (0,\infty)$.

Remark: Intuitively speaking, we could use a similar argument but with an interior maximum point. However, this implies that $u_t = 0, \Delta u \le 0$ but we might not have $u^2 - u = u(u-1) < 0$, since this requires knowing *a priori* (beforehand) that $u < 1$ is also true. A fix to this is first obtain $u(x,t) < 1$ via a perturbative argument, and once more to obtain $u(x,t) \le \kappa < 1$. However, this can get tedious and ugly real quick. A fix to this strategy is to observe that **since you want have to show that $u(x,t) \le \kappa < 1$, you want to be able to use this in $u^2 - u$ to show that $u^2 - u < 0$.** This is somewhat like an "induction over what you want to show", and the way to do this is to do a **first contact argument**. This is somewhat related to the continuity method/bootstrapping in more abstract PDE settings.

$u(x,t) \leq \kappa$ for all $(x,t) \in U \times (0,\infty)$ by a First Contact Argument.

Fix any $T > 0$. Denote the first contact time by

$$t_* = \min\{t \in [0,T] : \exists x \in \overline{U}, \quad u(x,t) = \kappa + \delta\} \tag{125}$$

for any $\delta \in (0, 1-\kappa)$. If we can show that $t_*$ does not exist, for any fixed $x \in U$, by the continuity of $u(x,t)$ in $t$, since it never crosses the line $u(x,t) = \kappa + \delta$ for any $t \in [0,T]$ and starts from $u(x,t) \leq \kappa$, by the Intermediate Value Theorem, we must have $u(x,t) < \kappa + \delta$ for each $t \in [0,T]$. Since $x \in U$ is arbitrary, we have that $u(x,t) < \kappa + \delta$. Since this argument is true for any $\delta > 0$, we can send $\delta \to 0$ and obtain $u(x,t) \leq \kappa$. Since this is true for all $t \in [0,T]$ for all $T > 0$, we have that it is true for all $t \in [0,\infty)$

We proceed by means of contradiction as follows. Suppose that $t_* < +\infty$. By the continuity of $u(\cdot, t_*)$ on the compact set $\overline{U}$ (since $U$ is bounded, so its closure is closed and bounded, thus compact), by the Extreme Value Theorem, there exists some $y_* \in \overline{U}$ such that $u(y_*, t_*)$ attains the maximum over $u(\cdot, t_*)$ for all $x \in \overline{U}$. Since $t_*$ is in the set in (125), there is some $x_* \in \overline{U}$ such that $u(x_*, t_*) = \kappa + \delta$.

Claim 1: $t_* > 0$. This follows from the fact if $t_* = 0$, then $u(x_*, 0) = \kappa + \delta$ for some $x_* \in \overline{U}$. However, for each $x \in \overline{U}$, we have $u(x,0) = g(x) \in [0,\kappa] < \kappa + \delta$, a contradiction.

Claim 2: $x_*$ must be the maximizer over $\overline{U}$ for a fixed $t = t_*$ (ie $y_* = x_*$). Proof: Suppose for a contradiction that this is not the case. Then there exists some $z_* \in \overline{U}$ such that $u(z_*, t_*) > \kappa + \delta$. By the continuity (and hence Intermediate Value Theorem since $u(z_*, 0) \leq \kappa$) of $u(z_*, \cdot)$ in time $t$, there exists some $s_* < t_*$ such that $u(z_*, s_*) = \kappa + \delta$, contradicting that $t_*$ is the infimum over the set as specified in (125). This claim thus proves that $u(x_*, t_*) = \kappa + \delta$ and is a maximizer over the spatial domain $\overline{U}$.

Claim 3: $\partial_t u(x_*, t_*) \geq 0$. Proof: By the definition of partial derivative from below, we have $\partial_t u(x_*, t_*) = \frac{u(x_*,t_*) - u(x_*,t_*-h)}{h}$. Since $u(x_*, t_*)$ is the first time in which it is $= \kappa + \delta$ and is a spatial maximizer, we must have $u(x_*, t_* - h) < u(x_*, t_*)$, and hence $\partial_t u(x_*, t_*) \geq 0$.

Hence, by the PDE, we have

$$\underbrace{u_t(x_*, t_*)}_{\geq 0} = \underbrace{\Delta u(x_*, t_*)}_{\leq 0} + \underbrace{u^2(x_*, t_*) - u(x_*, t_*)}_{<0} \tag{126}$$

since $u^2 - u < 0$ for $u = \kappa + \delta < 1$ (by our choice of $\delta \in (0, 1-\kappa)$), hence a contradiction. This implies that $t_* = \infty$, and by the first paragraph, we are done.

**Example 18.** (Spring 20, Problem 6, Elliptic.) If $U$ is the $n-$cube $\{x \in \mathbb{R}^n : -1 < x_i < 1 \quad \text{for each } i = 1, 2, \cdots, n\}$, show that the $C^2(U) \cap C(\overline{U})$ solutions to the equation

$$-\Delta u = -x \cdot \nabla u + 1 \tag{127}$$

depend continuously on the boundary data.

Specifically, show that if $\partial U$ is the boundary data of the cube, and $u_1$ and $u_2$ are solutions with $u_i(x)|_{\partial U} = g_i(x)$, then for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\|g_1 - g_2\|_{L^\infty} < \delta$ on $\partial U$, then $\|u_1 - u_2\|_{L^\infty} < \varepsilon$ in $\overline{U}$.[a]

[a]Note: $\|f\|_{L^\infty(U)} = \sup_{x \in U} |f(x)|$ for any $f \in L^\infty(U)$.

Suggested Solutions: Consider $u_i$ be the solutions to (127). Let $w = u_1 - u_2$. We observe that $w$ solves the following elliptic PDE:

$$\begin{cases} -\Delta w = -x \cdot \nabla w & \text{for } x \in U, \\ w(x)|_{\partial U} = g_1 - g_2. \end{cases} \tag{128}$$

Note: We would like to have the weak maximum principle hold for (128) to conclude. Observe that at a maximum for $w$, we have the following sign check

$$\underbrace{-\Delta w}_{\geq 0} = -x \cdot \underbrace{\nabla w}_{0},$$

which is almost a correct solution. This necessitates a perturbative argument.

Hence, we consider the perturbed $w_\varepsilon(x, t) := w(x, t) + \varepsilon\|x\|^2$. We then compute the derivatives as follows:

- $\nabla(w_\varepsilon) = \nabla w + 2\varepsilon x$, and
- $\Delta(w_\varepsilon) = \Delta w + 2n\varepsilon$.

This converts the elliptic PDE in (128) to

$$\begin{aligned} -\Delta w_\varepsilon + 2n\varepsilon &= -x \cdot (\nabla w_\varepsilon - 2\varepsilon x) \\ -\Delta w_\varepsilon &= -x \cdot \nabla w_\varepsilon - 2\varepsilon(\|x\|^2 - n). \end{aligned} \tag{129}$$

Let $x_*$ be point in $\overline{U}$ that maximizes $w_\varepsilon$ (which exists by Extreme Value Theorem). Suppose that $x_* \in U$. Hence, we have $\nabla w_\varepsilon = 0$ and $\Delta w_\varepsilon \leq 0$ at $x_*$. This implies that we have the following sign check:

$$\underbrace{-\Delta w_\varepsilon}_{\geq 0} = -x \cdot \underbrace{\nabla w_\varepsilon}_{0} \underbrace{-2\varepsilon(\|x\|^2 - n)}_{<0},$$

a contradiction. The last inequality can be observed from the fact that since $x$ is in the $n-$cube, we have each $|x_i| < 1$, $|x_i|^2 < 1$, and hence $\|x\|^2 = \sum_{i=1}^n |x_i|^2 < n$. Hence, we have the weak maximum principle for $w_\varepsilon$, that is,

$$\max_U w_\varepsilon \leq \max_{\partial U} w_\varepsilon.$$

In other words, the maximum is attained at the boundary of the set $U$.

Hence, for each $x \in U$, we have

$$w(x) = w_\varepsilon - \varepsilon\|x\|^2 \leq \max_U w_\varepsilon + 0 \leq \max_{\partial U} w_\varepsilon = \max_{\partial U}(w + \varepsilon\|x\|^2) \leq \max_{\partial U} w + \varepsilon \max_{\partial U} \|x\|^2 \leq \|g_1 - g_2\|_{L^\infty} + \varepsilon. \tag{130}$$

This then implies that

$$\|u_1 - u_2\|_{L^\infty} = \|w\|_{L^\infty} \leq \|g_1 - g_2\|_{L^\infty} + \varepsilon. \tag{131}$$

Sending $\varepsilon \to 0^+$, we have

$$\|u_1 - u_2\|_{L^\infty} \leq \|g_1 - g_2\|_{L^\infty}. \tag{132}$$

Hence, given any $\varepsilon > 0$, pick $\delta = \varepsilon$ such that $\|g_1 - g_2\|_{L^\infty} < \delta$. We then have by (132),

$$\|u_1 - u_2\|_{L^\infty} \leq \|g_1 - g_2\|_{L^\infty} < \delta = \varepsilon, \tag{133}$$

as required.

**Example 19.** (Fall 2018, Problem 4(i), Parabolic.) For a bounded domain $\Omega$ in $\mathbb{R}^n$ with smooth boundary, consider the parabolic PDE

$$\begin{cases} u_t - \Delta u = (1-u)_+ & \text{in } \Omega \times (0,\infty) \\ u(x,t) = l(x) & \text{in } \partial\Omega \times (0,\infty) \\ u(x,0) = g(x) & \text{in } \Omega, \end{cases} \tag{134}$$

with $(x)_+ := \max\{0, x\}$.

Show that if $l(x), g(x) \le 1$, then $u(x,t) \le 1$ for all $t > 0$.

Suggested Solutions: This appear to be like a problem requiring the use (and proof) of the maximum principles. Hence, we do a quick sign check[19] as follows:

$$\underbrace{u_t}_{\ge 0} - \underbrace{\Delta u}_{\le 0} = \underbrace{(1-u)_+}_{\ge 0}.$$

The signs are "completely wrong" as they go in the same direction. The possible fixes are as follows:

- Deactivate $(1-u)_+$ so it returns $0$.
- Use a perturbative argument to create a small positive term on the left.

By employing both fixes, we will arrive at a contradiction. To do so, we will employ them both in a single perturbative substitution below:

$$u_\varepsilon := u - 1 - \varepsilon t.$$

Hence, we have

- $u_t = (u_\varepsilon)_t + \varepsilon$,
- $\Delta u = \Delta(u_\varepsilon)$.

These imply

$$(u_\varepsilon)_t + \varepsilon - \Delta(u_\varepsilon) = (-u_\varepsilon - \varepsilon t)_+.$$

Fix any $T > 0$ and consider the parabolic boundary $\Gamma_T := (\overline{\Omega} \times \{t = 0\}) \cup (\partial\Omega \times [0,T])$. Suppose that $u_\varepsilon$ attains local maximum (exists on $\overline{\Omega} \times [0,T]$ by Extreme Value Theorem) at some $(x_*, t_*)$ not at the parabolic boundary $\Gamma_T$. This implies

- $(u_\varepsilon)_t(x_*, t_*) \ge 0$. If $t_* < T$, this must be zero (since it is a local maximum over time too). If $t_* = T$, this can only be $\ge 0$ since if $(u_\varepsilon)_t(x_*, t_*) < 0$, then $(u_\varepsilon)(x_*, s_*) < (u_\varepsilon)(x_*, t_*)$ for $s_* < t_*$ sufficiently close to $t_*$, contradicting that $(x_*, t_*)$ is a local maximum in time $t$ too.
- $\Delta u_\varepsilon(x_*, t_*) \le 0$ for a local maximum.

As mentioned above, we get the correct sign if $u_\varepsilon \ge 0$. Hence, at $(x_*, t_*)$, we have

- If $u_\varepsilon \ge 0$, then we have

  $$\underbrace{(u_\varepsilon)_t}_{\ge 0} + \varepsilon - \underbrace{\Delta(u_\varepsilon)}_{\le 0} = \underbrace{(-u_\varepsilon - \varepsilon t_*)_+}_{=0},$$

  a contradiction. This implies that $u_\varepsilon > 0$ obtains maximum at its parabolic boundary. This implies that

  $$\begin{aligned} u_\varepsilon(x,t) = u(x,t) - 1 - \varepsilon t &\le \max_{\Gamma_T} u_\varepsilon \le \max_{\Gamma_T} \left( \max\{l(x) - 1 - \varepsilon t, g(x) - 1\} \right) \\ &\le 0 \end{aligned} \tag{135}$$

  since $l(x), g(x) \le 1$.[20] This implies that

  $$u(x,t) \le 1 + \varepsilon T$$

  for all $(x,t) \in U \times [0,T]$. By sending $\varepsilon \to 0^+$, we have

  $$u(x,t) \le 1$$

  for all $(x,t) \in U \times [0,T]$. Since this is true for all $T > 0$, we can then deduce that

  $$u(x,t) \le 1 \quad \text{for all } (x,t) \in U \times (0,\infty). \tag{136}$$

[19] Recall that $u_t \ge 0$ if we are using a parabolic boundary formula, as it could be positive at $t = T$.

[20] Recall that $u_\varepsilon(x,t) = l(x) - 1 - \varepsilon t$ in $\partial\Omega \times (0,\infty)$ and $u_\varepsilon(x,0) = g(x) - 1$ in $\Omega$ by unpacking the substitution made.

- If $u_\varepsilon(x_*, t_*) < 0$, then $u_\varepsilon(x,t) \leq u_\varepsilon(x_*, t_*) < 0$ for all $(x,t) \in \overline{\Omega} \times [0,T]$ as $(x_*, t_*)$ corresponds to a local maximum. Unpacking the definition of $u_\varepsilon$, we have

$$u(x,t) = u_\varepsilon + 1 + \varepsilon t \leq 1 + \varepsilon t \leq 1 + \varepsilon T.$$

  Sending $\varepsilon \to 0^+$ and since this is true for all $T > 0$, we arrive at the same conclusion as in (136).

Remark: Note that this can also be done in a style similar to Example 17 for $u \geq 0$ by means of contradiction.

Qual problems for additional practice:

**Exercise 24.** (Fall 20, Problem 3, Elliptic.) Consider a smooth solution that satisfies the elliptic equation

$$\begin{cases} -\Delta u = (u+1)(u-1) & \text{in } \{|x|<1\} \\ u = f(x) & \text{on } \{|x|=1\}, \end{cases}$$

where $f$ is a continuous function such that $-1 < f(x) < 1$ on $|x| = 1$. Prove carefully that $-1 < u \leq -1$ in $|x| \leq 1$.

*Note: It is not sufficient to simply cite the maximum principle without reasoning.*

**Exercise 25.** (Spring 21, Problem 6, Parabolic.) Let $\Omega = \{|x| < 1\} \subset \mathbb{R}^n$ and for given continuous functions $g(x)$ and $u_0(x)$, let $u$ be a smooth solution of

$$\begin{cases} u_t - \Delta(u^2) = 0 & \text{in } \Omega \times (0,\infty) \\ u = g & \text{in } \partial\Omega \times (0,\infty) \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases}$$

Let $0 < g, u_0 < 1$. Show that $0 < u < 1$.

**Exercise 26.** (Fall 21, Problem 5, Elliptic.) A function $f(x,t)$ satisfies the nonlinear PDE

$$\Delta u - u^3 = 0$$

on a bounded, open domain $\Omega \subset \mathbb{R}^d$ with boundary conditions $u = g(x)$ on $\partial\Omega$. Assume that $u \in C^2(\Omega) \cup C(\overline{\Omega})$, and that $g(x) > 0$ at some $x \in \partial\Omega$.

(i) Show that $u(x) < \max_{X\in\partial\Omega}\{g(X)\}$ for all $x \in \Omega$.

(ii) Consider the special case for $d = 1$ and $\Omega = [-1,1]$. By choosing appropriate boundary conditions at $g(\pm 1)$, show that $u(x)$ can attain values less than $\min\{g(\pm 1)\}$.

**Exercise 27.** (Spring 22, Problem 3, Elliptic.) Suppose that $\mathbf{A}$ is symmetric, positive definite $n \times n$ matrix, $\mathbf{c}$ is a smooth vector field defined on a bounded open set $U \subset \mathbb{R}^n$, with piecewise smooth and orientable boundary $\partial U$. Show that the PDE:

$$-\sum_{i,j} A_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i c_i \frac{\partial u}{\partial x_i} = 0$$

with boundary conditions $u(x) = g(x)$ for $x \in \partial U$, has at most one solution that is $C^2(U) \cap C(\overline{U})$.

**Exercise 28.** (Spring 23, Problem 4, Parabolic.) Consider the nonlinear PDE

$$u_t = \Delta u - u^3, \quad x \in D, \quad 0 < t < T,$$

where $D \subset \mathbb{R}^n$ is a bounded and open set. You may assume that solutions exist and are $C^{2,1}(D\times(0,T)) \cap C(\overline{D} \times [0,T])$. Show that the solutions of the PDE are unique.

# 8 Discussion 8

An Introduction to Calculus of Variation.
(Evans Chapter 2.2 and Chapter 8.)

Consider the boundary-value Poisson's equation:

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{137}$$

where $u \in C^2(U) \cap C(\overline{U})$ and $f \in C(\overline{U})$, for some open bounded subset $U$ of $\mathbb{R}^n$ with a piecewise smooth boundary.

A key concept for elliptic equations is that solution to these equations usually corresponds to the minimizer of an appropriate functional. To see this, we consider the following *energy functional*:

$$E[w] := \int_U \left( \frac{1}{2} |\nabla w|^2 - wf \right) \mathrm{d}x \tag{138}$$

with $w$ belonging to some *admissible set* given by

$$\mathcal{A} := \{ w \in C^2(U) \cap C(\overline{U}) | w = g \text{ on } \partial U \}. \tag{139}$$

The following theorem illustrates this connection.

**Theorem 20.** (Dirichlet's Principle.) Assume that $u \in C^2(U) \cap C(\overline{U})$. Then $u$ solves (137) if and only if $u \in \mathcal{A}$ and

$$E[u] = \min_{w \in \mathcal{A}} E[w] \tag{140}$$

with $E[\cdot]$ defined in (138).

The proof of Theorem 20 is important, as it generalizes to other energy functionals. Furthermore, (140) is also known as the **minimizing principle** in qual problems.

Before we begin the proof, we supplement this with a lemma from real analysis as follows:

**Lemma 21.** (Fundamental Lemma of Calculus of Variation.) Let $U$ be an open subset of $\mathbb{R}^n$ and $f \in C(U)$. If

$$\int_U f\varphi \, \mathrm{d}x = 0 \quad \text{for all } \varphi \in C_c^\infty(U),$$

then $f \equiv 0$ in $U$.

This lemma can be proven using the Lebesgue differentiation theorem and Dominated convergence theorem, which is beyond the scope of this class.

**Proof of Theorem 20:** ( $\Longrightarrow$ ) To show that (140), we want to show that for any given $w \in \mathcal{A}$, we have

$E[u] \leq E[w]$. Equivalently, we want to show that $E[u] - E[w] \leq 0$. Hence, we compute it as follows:

$$
\begin{aligned}
E[u] - E[w] &= \int_U \frac{1}{2}\left(|\nabla u|^2 - |\nabla w|^2\right) - f(u-w)\mathrm{d}x \\
&\underbrace{=}_{(137)} \int_U \frac{1}{2}\left(|\nabla u|^2 - |\nabla w|^2\right) + \Delta u(u-w)\mathrm{d}x \\
&= \int_U \frac{1}{2}\left(|\nabla u|^2 - |\nabla w|^2\right) + \nabla\cdot(\nabla u(u-w)) - \nabla u \cdot \nabla(u-w)\mathrm{d}x \\
&= \int_U \frac{1}{2}\left(|\nabla u|^2 - |\nabla w|^2\right) - |\nabla u|^2 + \nabla u\cdot\nabla w\mathrm{d}x + \int_{\partial U} \underbrace{(u-w)}_{u|_{\partial U} = w|_{\partial U} = g \implies (u-w)|_{\partial U} = 0} \nabla u\cdot\mathbf{n}\,\mathrm{d}S \\
&= \frac{1}{2}\int_U -|\nabla w|^2 - |\nabla u|^2 + 2\nabla u\cdot\nabla w\mathrm{d}x \\
&\leq 0
\end{aligned}
\tag{141}
$$

where we have used the fact that

$$2\nabla u\cdot\nabla w \leq \|\nabla u\|^2 + \|\nabla w\|^2$$

by Cauchy Schwarz.

Since this is true for all $w \in \mathcal{A}$ and $u \in \mathcal{A}$, we then deduce (140).

$(\Longleftarrow)$ To show that $u$ solves the PDE (137), we proceed as follows. Fix any $v \in C_c^\infty(U)$ (the space of smooth functions with compact support in $U$) and consider

$$e(\varepsilon) := E[u + \varepsilon v].$$

Note that since $v \in C_c^\infty(U)$, then we must have $u + \varepsilon v \in C^2(U) \cap C(\overline{U})$ with $u + \varepsilon v|_{\partial U} = g$ (since $v|_{\partial U} = 0$). Since $E[u] \leq E[u + \varepsilon v]$ for any $\varepsilon \in \mathbb{R}$, then the function $e(\varepsilon)$ attains a global minimum at $\varepsilon = 0$. This then implies that $e'(0) = 0$.

To use this relation, we first compute $e(\varepsilon)$ as follows:

$$
\begin{aligned}
e(\varepsilon) &= E[u + \varepsilon v] \\
&= \int_U \frac{1}{2}|\nabla(u + \varepsilon v)|^2 - (u + \varepsilon v)f\mathrm{d}x \\
&= \int_U \frac{1}{2}(|\nabla u|^2 + 2\varepsilon\nabla u\cdot\nabla v + \varepsilon^2|\nabla v|^2) - uf - \varepsilon vf\mathrm{d}x \\
&= \int_U \frac{1}{2}|\nabla u|^2 - uf\mathrm{d}x + \varepsilon\left(\int_U \nabla u\cdot\nabla v - vf\mathrm{d}x\right) + \frac{\varepsilon^2}{2}\int_U |\nabla v|^2\mathrm{d}x.
\end{aligned}
\tag{142}
$$

We can then compute $e'(\varepsilon)$ to obtain

$$e'(\varepsilon) = \int_U \nabla u\cdot\nabla v - vf\mathrm{d}x + \varepsilon\int_U |\nabla v|^2\mathrm{d}x \tag{143}$$

and thus $e'(0) = 0$ implies

$$\int_U \nabla u\cdot\nabla v - vf\mathrm{d}x = 0. \tag{144}$$

Integrating by parts (divergence theorem) and using the fact that $v|_{\partial U} = 0$ since it has compact support, we deduce that

$$\int_U (-\Delta u - f)v\mathrm{d}x = 0 \quad \text{for all } v \in C_c^\infty(U). \tag{145}$$

Since (145) is true for all $v \in C_c^\infty(U)$, by the Fundamental Lemma for Calculus of Variation, we deduce that

$$-\Delta u = f \quad \text{in } U. \tag{146}$$

Furthermore, since $u \in \mathcal{A}$, we also have $u = g$ on $\partial U$. $\square$

Remark: Note that $\varepsilon = 0$ is a minimizer of $e(\varepsilon)$ is somewhat reflected in (143), in which we can compute $e''(0)$ to obtain

$$e''(0) = \int_U |\nabla v|^2 \mathrm{d}x,$$

which is positive as long as $|\nabla v| \neq 0$ on a set of positive measure. If that is not the case, then $|\nabla v| = 0$ almost everywhere, and thus $v$ is a constant function almost everywhere in $U$. Since $v$ vanishes in a compact subset of $U$, this constant must be zero. If this is the case, then we are just looking at the "zero perturbation to the energy functional", and it is expected that $e(\varepsilon)$ is a constant, which further implies that we are essentially not utilizing the information that $u$ minimizes the energy functional enough.

Remark: From the above computations, we obtained

$$e'(0) = \int_U (-\Delta u - f) v \mathrm{d}x.$$

In the calculus of variation literature, $e'(0)$ is referred to the **Gateaux derivative** of the energy function $E[\cdot]$ at $u$ in the direction $v$. This concept generalizes the concept of directional derivatives in calculus.

Next, we will illustrate that we can use the minimizing principle, ie (140), to prove uniqueness of solutions to (elliptic) PDEs. By Theorem 20, to show that solutions to (137) is unique, it suffices to show that there is a unique minimizer for the energy functional $E[\cdot]$ in (140). To do so, we proceed as follows.

Let $u, v$ be two minimizers of $E[\cdot]$. Observe that

$$\begin{aligned} E\left[\frac{u+v}{2}\right] &= \int_U \frac{1}{2}\left|\nabla\left(\frac{u+v}{2}\right)\right|^2 - f\left(\frac{u+v}{2}\right)\mathrm{d}x \\ &= \int_U \frac{1}{8}|\nabla u|^2 + \frac{1}{4}\nabla u\cdot\nabla v + \frac{1}{8}|\nabla v|^2 - \frac{1}{2}fu - \frac{1}{2}fv\mathrm{d}x \\ &= \int_U \frac{1}{4}|\nabla u|^2 + \frac{1}{4}|\nabla v|^2 - \frac{1}{8}|\nabla u - \nabla v|^2 - \frac{1}{2}fu - \frac{1}{2}fv\mathrm{d}x \\ &\le \int_U \frac{1}{4}|\nabla u|^2 + \frac{1}{4}|\nabla v|^2 - \frac{1}{2}fu - \frac{1}{2}fv\mathrm{d}x \\ &\le \frac{1}{2}(E[u]+E[v]) \le \min_{w\in\mathcal{A}} E[w]. \end{aligned} \tag{147}$$

Observe that the first inequality is strict if $(|\nabla u| - |\nabla v|)^2 \neq 0$ on a set of positive measure. This implies that $\frac{u+v}{2}$ attains a lower energy, and thus contradicting that $u$ and $v$ are minimizers of the energy functional.

Suppose instead that $(|\nabla u| - |\nabla v|)^2 = 0$ (almost everywhere) in $U$. This implies that $\nabla u = \nabla v$ in $U$. Hence, we deduce that $u = v + C$ for some constant $C$. Since $u, v \in \mathcal{A}$, hence $u|_{\partial U} = v|_{\partial U} = g$, which then implies that $g = g + C$ and thus $C = 0$. Hence, we deduce that $u = v$, thus obtaining the required uniqueness claim.

Remark: In fact, the whole argument can be summarized by saying that if the energy function is **strictly convex**, then $E[\cdot]$ has a unique minimizer. Here, a functional $E[\cdot]$ is strictly convex if for all $u \neq v \in \mathcal{A}$ and $\lambda \in (0,1)$, we have

$$E[\lambda u + (1-\lambda)v] < \lambda E[u] + (1-\lambda)E[v].$$

Remark: For more advanced concepts in calculus of variations, such as constrained optimization, Euler-Lagrange equations, see Evans Chapter 8 or take Math 273B (for a more applied perspective of these).

**Example 22.** (Spring 23, Problem 3, Elliptic.) Consider the energy functional $E[u]$, which is defined for all $u \in C^2(\mathcal{D}) \cap C(\overline{\mathcal{D}})$ by

$$E[u] = \frac{1}{2}\int_{\mathcal{D}} (|\nabla u|^2 + u^2)\mathrm{d}x$$

where $\mathcal{D} \subset \mathbb{R}^n$ is a bounded and open set. Assume that $u = g(x)$ is known on $\partial D$.

(i) Derive the PDE satisfied by minimizers of $E[\cdot]$.

(ii) Starting from the minimization principle, prove that solutions to the PDE are unique.

(iii) Suppose that $n = 1, \mathcal{D} = (-1, 1)$, and $u(-1) = u(1) = 1$. Find an approximate solution of your PDE from (i) that takes the form $u = 1 + A(1 - x^2)$. In other words, find the value of $A$ that minimizes the energy functional.

Suggested Solutions:

(i) Define the admissible class by

$$\mathcal{A} := \{w \in C^2(\mathcal{D}) \cap C(\overline{D}) | u = g \text{ on } \partial\mathcal{D}\}.$$

If $u$ minimizes $E$ over $\mathcal{A}$, take any $v \in C_c^\infty(\mathcal{D})$ and consider

$$e(\varepsilon) := E[u + \varepsilon v].$$

This implies that $e'(0) = 0$. To utilize this relation, we first compute $e'(\varepsilon)$ as follows:

$$\begin{aligned} e(\varepsilon) &= \frac{1}{2}\int_{\mathcal{D}} |\nabla(u + \varepsilon v)|^2 + (u + \varepsilon v)^2 \mathrm{d}x \\ &= \frac{1}{2}\left(\int_{\mathcal{D}} |\nabla u|^2 + u^2 \mathrm{d}x\right) + \varepsilon\left(\int_{\mathcal{D}} \nabla u \cdot \nabla v + uv\mathrm{d}x\right) + \frac{\varepsilon^2}{2}\left(\int_{\mathcal{D}} |\nabla v|^2 + v^2 \mathrm{d}x\right). \end{aligned} \tag{148}$$

Hence, we have

$$e'(0) = \int_{\mathcal{D}} \nabla u \cdot \nabla v + uv\mathrm{d}x = 0 \tag{149}$$

and by applying divergence theorem and $v|_{\partial\mathcal{D}} = 0$ since it has compact support, we obtain

$$\int_{\mathcal{D}} (-\Delta u + u)\, v\mathrm{d}x = 0 \quad \text{for all } v \in C_c^\infty(\mathcal{D}). \tag{150}$$

Since this is true for all $v \in C_c^\infty(\mathcal{D})$, by the Fundamental Lemma for Calculus of Variation, we have

$$-\Delta u + u = 0 \quad \text{in } U.$$

Furthermore, since $u \in \mathcal{A}$, we also have $u|_{\partial\mathcal{D}} = g$. Henceforth, the PDE satisfied by minimizers of $E[\cdot]$ is given by

$$\begin{cases} -\Delta u + u = 0 & \text{in } \mathcal{D}, \\ u = g & \text{on } \partial\mathcal{D}. \end{cases} \tag{151}$$

(ii) Note that (i) tells us that each minimizer of $E[\cdot]$ satisfies the PDE (151). Here, we assume that we have the other direction (ie solutions to the PDE are minimizers of $E[\cdot]$ (this is what I assume by "starting from the minimizing principle"). Hence, if we have a unique minimizer of $E[\cdot]$, then we have a unique solution to the PDE.

To show that we have a unique minimizer, let $u$ and $v$ be minimizers of $E[\cdot]$. Then, observe that

$$
\begin{aligned}
E\left[\frac{u_1+u_2}{2}\right] &= \frac{1}{8}\int_{\mathcal{D}}\left(|\nabla u+\nabla v|^2+(u+v)^2\right)\mathrm{d}x \\
&= \frac{1}{8}\int_{\mathcal{D}} |\nabla u|^2 + 2\nabla u\cdot\nabla v + |\nabla v|^2 + u^2 + 2uv + v^2 \mathrm{d}x \\
&= \frac{1}{8}\int_{\mathcal{D}} 2|\nabla u|^2 + 2|\nabla v|^2 - |\nabla u - \nabla v|^2 + 2u^2 + 2v^2 - (u-v)^2 \mathrm{d}x \\
&\leq \frac{1}{8}\int_{\mathcal{D}} 2|\nabla u|^2 + 2|\nabla v|^2 + 2u^2 + 2v^2 \mathrm{d}x \\
&= \frac{1}{2}E[u] + \frac{1}{2}E[v] \leq \min_{w\in\mathcal{A}} E[w].
\end{aligned}
\tag{152}
$$

Observe that the first inequality is strict if $(u-v)^2 \neq 0$ on a set of positive measure, which implies that $\frac{u+v}{2}$ attains a lower energy, thus contradicting that $u$ and $v$ are minimizers of the energy functional.

If $(u-v)^2 = 0$ almost everywhere, then $u = v$, implying that we have a unique minimizer.

(iii) Since $u = 1 + A(1-x^2)$, then $u'(x) = -2Ax$. Hence, the energy functional evaluated with this ansatz is given by

$$E[u] = \frac{1}{2}\int_{-1}^{1} (-2Ax)^2 + (1 + A(1-x^2))^2 \mathrm{d}x.$$

This simplifies to

$$E[u] = \frac{1}{2}\int_{-1}^{1} (1+A)^2 + 2A(A-1)x^2 + A^2x^4 \mathrm{d}x$$

and hence

$$E[u] = 1 + \frac{4}{3}A + \frac{28}{15}A^2.$$

As an approximate solution to the PDE, we minimize the energy with respect to this parameter $A$. This is thus obtained at $A = -\frac{5}{14}$, and the approximate solution to the PDE is given by

$$u(x) \approx 1 - \frac{5}{14}(1-x^2).$$

Qual problems for additional practice:

**Exercise 29.** (Fall 18, Problem 6, Elliptic.) Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and consider

$$u \in \mathcal{A} := \{u \in C^1(\Omega) | u = 0 \text{ on } \partial\Omega, \text{ and } \int_\Omega u = 1\}$$

and consider the energy functional

$$E(u) := \int_\Omega \sqrt{1 + |\nabla u|^2(x)} \mathrm{d}x.$$

(i) Show that $E(u)$ has at most one minimizer among $u \in \mathcal{A}$.

(ii) Let $\Omega := \{|x| < 1\}$ and suppose that $u^*$ minimizes $E(u)$ in $\mathcal{A}$. Show that $u$ is a radial function.

**Exercise 30.** (Spring 19, Problem 8, Elliptic.) Let $u$ be a solution of Poisson's equation in a domain $U$:

$$-\Delta u = f$$

for some smooth function $f(x)$. The piecewise smooth boundary of $U$ can be divided into sets with measure $0$ intersection, we call these $\partial U_N$ and $\partial U_D$. On $\partial U_N$, the normal derivative $\frac{\partial u}{\partial n} = N(x)$ is known, whereas on $\partial U_D$, $u(x) = h(x)$ is prescribed.

(i) Show that $u(x)$ minimizes the functional:

$$E[u] = \int_U \frac{1}{2}|\nabla u|^2 - f(u)\mathrm{d}V - \int_{\partial U_N} N u \mathrm{d}s$$

among a set of $C^2$ functions that you should identify.

(ii) Suppose that we are studying flow in a pipe whose cross-section is the unit square $|x|, |y| < 1$. The flow field solves the Poisson equation:

$$-\Delta u = 1.$$

On three of the walls of the pipe: $x = -1$ and $y = \pm 1$, we have $u = 0$. On the fourth wall $(x = 1)$, we have $\frac{\partial u}{\partial x} = 0$. Explain how you could use the minimization principle from (a) to calculate optimal constants $A$, $x_1, x_2, y_1, y_2$ in an approximation to the flow field of the form:

$$u(x, y) = A(x_1 - x)(x_2 - x)(y_1 - y)(y_2 - y).$$

(You don't need to evaluate your integrals to calculate $A$ explicitly, but you should identify the constants $x_1, x_2, y_1,$ and $y_2$.)

**Exercise 31.** (Spring 21, Problem 7, Elliptic.) Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and consider

$$u \in \mathcal{A} := \{u \in C^1(\Omega) | u = 0 \text{ on } \partial\Omega, \text{ and } \int_\Omega u = 1\}$$

and consider the energy functional

$$E(u) := \int_\Omega \sqrt{1 + |\nabla u|^2(x)} + |x|^2 u(x) \mathrm{d}x.$$

(i) Show that $E(u)$ has at most one minimizer among $u \in \mathcal{A}$.

(ii) Let $\Omega := \{|x| < 1\}$ and suppose that $u^*$ minimizes $E(u)$ in $\mathcal{A}$. Show that $u$ is a radial function.

# 9 Discussion 9

Fundamental Solutions to the Laplace and Poisson's Equations.
(Shearer and Levy Chapters 8.1 and 8.2, Evans Chapter 2.2.1)

Consider the Laplace equation:

$$-\Delta u = 0 \quad \text{in } \mathbb{R}^n.$$

The **fundamental solution** is given by

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi}\log|x| & (n=2) \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & (n \geq 3), \end{cases}$$

with $\alpha(n)$ representing the volume of a unit ball in $\mathbb{R}^n$. Note that $\Phi$ satisfies the Laplace equation **except at** $x=0$, in which by direct computation, one could observe a singularity for $\Delta\Phi$ at $x=0$. Henceforth, in the sense of distributions, we have

$$-\Delta\Phi(x) = \delta(x),$$

where $\delta(x)$ refers to the "Dirac delta function", or in rigorous mathematical terms - a Dirac measure on $\mathbb{R}^n$ with a unit mass at the point $0$ (denoted by $\delta_0$ in this case).

Here, $\delta(x)$ (formally) has the following properties:

- $\int_U \delta(x)\mathrm{d}x = 1$ for $U \subset \mathbb{R}^n$ containing $0$,
- $\int_U f(x)\delta(x)\mathrm{d}x = f(0)$ for $U \subset \mathbb{R}^n$ containing $0$, and
- $\int_U f(x)\delta(x-y)\mathrm{d}x = f(y)$ for $U \subset \mathbb{R}^n$ containing $y$.

These properties can be proved rigorously by viewing $\delta(x) = \delta_0$ and $\delta(x-y) = \delta_y$ as distributions and applying the definition of distributions.

Next, we can construct the solution to the corresponding Poisson's equation:

$$-\Delta u = f \quad \text{in } \mathbb{R}^n$$

with a sufficiently regular $f$, ie $f \in C_c^2(\mathbb{R}^n)$.[21] The **solution to the Poisson equation in** $\mathbb{R}^n$ is given by

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\mathrm{d}y.$$

We can see that this is formally true as follows:

$$-\Delta u(x) = \int_{\mathbb{R}^n} -\Delta\Phi(x-y)f(y)\mathrm{d}y = \int_{\mathbb{R}^n} \delta(x-y)f(y)\mathrm{d}y = f(x)$$

for each $x \in \mathbb{R}^n$.

[21] $C_c^2(\mathbb{R}^n)$ refers to a twice continuously differentiable function with compact support.

Green's Function for Poisson's Equation.
(Shearer and Levy Chapters 9.3 and 9.4, Evans Chapter 2.2.4.)
(Remark: We do not cover Green's function in 1D since this has been extensively covered in 266A.)

Consider the boundary-value Poisson's equation:

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{153}$$

where $u \in C^2(\overline{U})$, $f$ and $g$ sufficiently smooth, for some open bounded subset $U$ of $\mathbb{R}^n$ with $C^1$ boundary.

Recall the Green's identity as follows (which can be easily derived using divergence theorem): for any $u(x), v(x) \in C^2(\overline{U})$, we have[22]

$$\int_U u(x)\Delta v(x) - v(x)\Delta u(x)\mathrm{d}x = \int_{\partial U} \left( u(x)\frac{\partial v}{\partial \nu}(x) - v(x)\frac{\partial u}{\partial \nu}(x) \right) \mathrm{d}S(x).$$

For any two functions $u(x) \in C^2(\overline{U})$ and $G(x,y)$ (for a fixed $y$) on $U$, we have

$$\int_U u(x)\Delta_x G(x,y) - G(x,y)\Delta u(x)\mathrm{d}x = \int_{\partial U} u(x)\frac{\partial G}{\partial \nu_x}(x,y) - G(x,y)\frac{\partial u}{\partial \nu}(x)\mathrm{d}S(x). \tag{154}$$

The idea of a Green's function is to pick the appropriate $G(x,y)$ such that (154) yields a valid representation of the solution. For instance, if we pick $G(x,y) = \Phi(x-y)$ where $\Phi$ is the fundamental solution to the Laplace equation, then we have $-\Delta_x G(x,y) = \delta(x-y)$.[23] This implies that for any fixed $y \in U$, $G(x,y) = \Phi(x-y)$ on $x \in \partial U$, which is not necessarily zero. This implies in (154), together with the fact that $\int_U u(x)\delta(x-y)\mathrm{d}x = u(y)$, we formally[24] have

$$\begin{aligned} u(y) &= \int_U \Phi(x-y)\Delta u(x)\mathrm{d}x + \int_{\partial U} u(x)\frac{\partial \Phi}{\partial \nu_x}(x-y) - \Phi(x-y)\frac{\partial u}{\partial \nu}(x)\mathrm{d}S(x) \\ &= \int_U G(x,y)\Delta u(x)\mathrm{d}x + \int_{\partial U} u(x)\frac{\partial G}{\partial \nu_x}(x,y) - G(x,y)\frac{\partial u}{\partial \nu}(x)\mathrm{d}S(x) \end{aligned} \tag{155}$$

Employing (155) as the representative formula for $u$ is okay only if the value of $u$ and $\frac{\partial u}{\partial \nu}$ are both known, which is usually not the case. (In fact, having both imposed might sometimes lead to lack of solutions or possibly uniqueness.) Note that the second equality in (155) is valid as long as $-\Delta_x G(x,y) = \delta(x-y)$ for any choice of $G$.

Henceforth, the goal is to use (155) to complement the corresponding PDE to be solved. In particular, we need either $\frac{\partial G}{\partial \nu_x}(x,y)$ or $G(x,y)$ to vanish the boundary $\partial U$ for each given $y \in U$ if we do not have the corresponding information for $u$ or $\frac{\partial u}{\partial \nu}$ respectively. Furthermore, for the integral $\int_U u(x)\Delta_x G(x,y)\mathrm{d}x$ to turn into $u(y)$, we also require $-\Delta_x G(x,y) = \delta(x-y)$.

As an example, to solve (153), for us to employ (155) as the correct representation, we need

$$-\Delta_x G(x,y) = \delta(x-y).$$

Now, with (155) as the representation, note that we are given $u = g$ on $\partial U$. What we are **not given** is $\frac{\partial u}{\partial \nu}$ on $U$. Henceforth, by the representation formula in (155), we want the term containing $\frac{\partial u}{\partial \nu}$ to vanish. To do so, we set $G(x,y) = 0$ for $x \in \partial U$ (this is to hold for each fixed $y \in U$ from the start of the argument). Hence, for any fixed/given $y \in U$, the Green's function for (153) must solve

$$\begin{cases} -\Delta_x G(x,y) = \delta(x-y) & x \in U, \\ G(x,y) = 0 & x \in \partial U. \end{cases} \tag{156}$$

The following example illustrates the above concept, and how one can go about computing the relevant Green's function.

[22] $\frac{\partial}{\partial \nu}$ refers to the directional derivative normal to the surface.
[23] Recall that from the previous section on distributions, for a fixed $y$, we can view $\delta(x-y)$ as a unit mass at $y$. Alternatively, for practical purposes, we can "apply" this from a physical point of view.
[24] To derive this rigorously, we have to cut the open set up with a ball of size $\varepsilon$ centered at $y$. For more information, refer to Evans Chapter 2.2.4.

**Example 23.** Consider $U = \{x \in \mathbb{R}^2 : x_2 > 0\}$ and the Neumann problem for the Poisson's equation:

$$\begin{cases} -\Delta u = f & \text{in } U, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial U. \end{cases} \tag{157}$$

Furthermore, we assume that $f$ and $g$ have compact support, and $f$ and $g$ are sufficiently regular such that solutions to (157) exists.[a]

(i) Prove that there is only at most one $C_c^2(\overline{U})$ solution to (157).

(ii) Compute the expression for the Green's function for the Neumann problem in (157) explicitly.

[a] Proving existence of solutions to elliptic problems are outside the scope of 266B; for more information, see 266C/269C.

Suggested Solutions:

(i) Let $u$ and $v$ be solutions to (157) and let $w := u - v$. Then, we have that $w$ solves

$$\begin{cases} -\Delta w = 0 & \text{in } U, \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial U, \end{cases} \tag{158}$$

with $w \in C_c^2(\overline{U})$.

By considering the following energy, we have[25]

$$\begin{aligned} 0 &\leq \int_U |\nabla w|^2 \mathrm{d}x \\ &= \int_U \nabla \cdot (w \nabla w) - w \Delta w \mathrm{d}x \\ &= \int_{\partial U} w \frac{\partial w}{\partial n}(x) \mathrm{d}S + \int_U w \Delta w \mathrm{d}x \\ &= 0. \end{aligned} \tag{159}$$

This implies that $\int_U |\nabla w|^2 \mathrm{d}x = 0$, and hence $\nabla w = 0$ in $U$. This in turn implies that $w = C$ for some constant $C$ in $U$. Since $w \in C_c^2(\overline{U})$, by continuity up to its compact support (at which $w$ soon vanishes), we deduce that $C = 0$. Hence $u = v$ and the uniqueness claim follows.

(ii) Suppose that we have

$$-\Delta_x G(x, y) = \delta(x - y),$$

then by the representation formula in (155), we have

$$u(y) = \int_U G(x, y) \Delta u(x) \mathrm{d}x + \int_{\partial U} u(x) \frac{\partial G}{\partial \nu_x}(x, y) - G(x, y) \frac{\partial u}{\partial \nu}(x) \mathrm{d}S(x). \tag{160}$$

Since $\frac{\partial u}{\partial \nu}$ but not $u$ is given in $\partial U$, we would like the term associated with $u$ to vanish. To do so, we demand that for any given $y \in U$, we have

$$\frac{\partial G}{\partial \nu_x}(x, y) = 0 \quad \text{for all } x \in \partial U.$$

Hence, we would like our Green's function to satisfy

$$\begin{cases} -\Delta_x G(x, y) = \delta(x - y) & x \in U, \\ \dfrac{\partial G}{\partial \nu_x}(x, y) = 0 & x \in \partial U. \end{cases} \tag{161}$$

Recall that picking $G(x, y) = \Phi(x - y)$ satisfies $-\Delta_x G(x, y) = \delta(x - y)$, but $\frac{\partial G}{\partial \nu_x}(x, y) = 0$ is not necessarily satisfied along $x \in \partial U$.

[25] Note that the expressions here make sense since $\|w\|_{C_c^2(\overline{U})} < +\infty$ since it has compact support. (For unbounded domains, we do not want to have $\infty \times 0$ case happening and that the expressions we are manipulating are invalid since we are manipulating infinites.)

In fact, one can observe by direct computation that in $\mathbb{R}^2$, since $\Phi(x-y) = -\frac{1}{2\pi}\log|x-y|$, we have

$$\nabla_x \Phi(x-y) = -\frac{1}{2\pi}\frac{1}{|x-y|^2}\begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix}.$$

Since $\mathbf{n} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ along $x_2 = 0$ (ie $\partial U$), we have for each $x_1 \in \mathbb{R}$ (with $G(x,y) := \Phi(x-y)$),

$$\frac{\partial G}{\partial \nu_x}((0,x_2),y) = \nabla_x G \cdot \mathbf{n}((x_1,0),y) = -\frac{1}{2\pi}\frac{1}{|(x_1,0)-(y_1,y_2)|^2}\begin{pmatrix} x_1 - y_1 \\ 0 - y_2 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\frac{y_2}{2\pi|(x_1,0)-(y_1,y_2)|^2}. \tag{162}$$

This is not true since $y_2 \neq 0$ (recall that $y \in U$).

Henceforth, we would like an additional term with the **same sign** as $\Phi(x-y)$, but at $\tilde{y} := (y_1, -y_2)$. In other words, if we let

$$G(x,y) := \Phi(x-y) + \Phi(x-\tilde{y}), \tag{163}$$

then

$$-\Delta_x G(x,y) = \delta(x-y) + \delta(x-\tilde{y}).$$

Note that since $\tilde{y} \notin U$, we have that $\delta(x-\tilde{y})$ is always zero for each $x \in U$ and thus

$$-\Delta_x G(x,y) = \delta(x-y). \tag{164}$$

Henceforth, the first requirement for (161) is satisifed. To check that it satisfies the Neumann boundary condition along $x_2 = 0$, observe that from our computations in (162), we have

$$\begin{aligned} \frac{\partial G}{\partial \nu_x}((0,x_2),y) &= \nabla_x G \cdot \mathbf{n}((x_1,0),y) \\ &= -\frac{y_2}{2\pi|(x_1,0)-(y_1,y_2)|^2} + -\frac{-y_2}{2\pi|(x_1,0)-(y_1,-y_2)|^2} \\ &= -\frac{y_2}{2\pi|(x_1,0)-(y_1,y_2)|^2} + \frac{y_2}{2\pi|(x_1,0)-(y_1,y_2)|^2} \\ &= 0. \end{aligned} \tag{165}$$

Here, we have indicated the contribution of $\Phi(x-\tilde{y})$ on the second term in the second equality in (165), and use the fact that $|(x_1,0)-(y_1,-y_2)| = \sqrt{(x_1-y_1)^2 + (-y_2)^2} = \sqrt{(x_1-y_1)^2+(y_2)^2} = |(x_1,0)-(y_1,y_2)|$.

Henceforth, the Green's function for (157) is given in (163).

Remark: There is a more "intuitive" way to solve for the Green's function satisfying (161) if you can map this problem into an **electrostatics problem**. For instance, the governing equations for electrostatics are given by

$$\begin{aligned} \nabla \cdot E &= \frac{\rho}{\varepsilon_0}, \\ \nabla \times E &= 0. \end{aligned} \tag{166}$$

Here, $E$ represents the electric field, $\rho$ represents the charge density, and $\varepsilon_0$ represents a physical constant (called vacuum permittivity). The second equation implies that the electric field $E$ is conservative, and hence there is an eletric potential $V$ such that $-\nabla V = E$. Plugging this to the first equation, we obtain

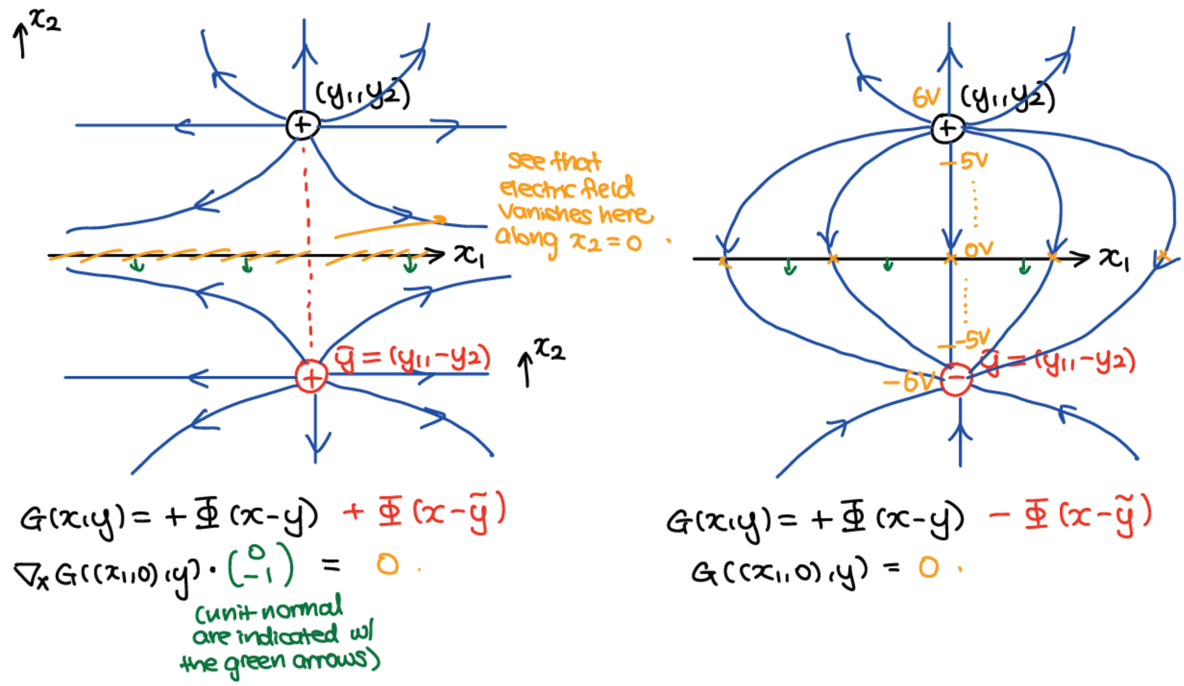
$$-\Delta V = \frac{\rho}{\varepsilon_0}. \tag{167}$$

This is precisely the Poisson's equation, with $\frac{\rho}{\varepsilon_0}$ as the charge density (normalized by a constant). Henceforth, the physical interpretation of

$$-\Delta V(x) = \delta(x-y) \tag{168}$$

refers obtaining the electric potential in space corresponding to a **positive point charge** at $y$ (for a given $y$). The condition of requiring $\frac{\partial V}{\partial \nu} = \nabla V \cdot \mathbf{n} = -E \cdot \mathbf{n}$ to vanish along a boundary is equivalent to **requiring the electric field to vanish along the boundary**. In our example for which we already have a positive point charge at $y$, the electric field along $x_2 = 0$ vanishes only if we have a positive charge at $\tilde{y} := (y_1, -y_2)$.

Similarly, for the Dirichlet problem, if we would like the potential to vanish along the boundary, using the same domain $U := \{x_2 > 0\}$, a positive test charge at $y$ needs to have a negative test charge at $\tilde{y} := (y_1, -y_2)$ to have the potential to vanish.[26]




In summary,

- For the Green's function to vanish at a boundary, do an "odd reflection" across the boundary. i.e **Subtract** $\Phi(x - \tilde{y})$, where $\tilde{y}$ is obtained upon reflection.
- For the normal derivative of the Green's function to vanish at the boundary, do an "even reflection" across the boundary. i.e **Add** $\Phi(x - \tilde{y})$, where $\tilde{y}$ is obtained upon reflection.

[26] Note that electric potential has the issue that if $V(x)$ is an electric potential, then $V(x) + C$ is an electric potential for any constant $C > 0$. Hence, strictly speaking, we are usually looking at an equivalent problem with $C = 0$. This is consistent with physical conventions that the potential as $|x| \to \infty$ is usually set to be zero.

Distributions and Weak Solutions.
(Shearer and Levy Chapter 9.2, Evans Chapter 3.4.1.)

Recall from Shearer and Levy Chapter 9.2 that

- $\mathcal{D} := C_c^\infty(\mathbb{R}^n)$ is the space of compactly supported smooth functions in $\mathbb{R}^n$ (a.k.a **test functions**).
- $\mathcal{D}'$ is defined as the **space of distributions**. In other words, each $f \in \mathcal{D}'$ is a **distribution**.
- A distribution $f$ is a linear continuous functional on $\mathcal{D}'$ (ie $f : \mathcal{D} \to \mathbb{R}$, ie $f(\phi) \in \mathbb{R}$ for each $\phi \in \mathcal{D}$). In other words, it has the follwing properties:
  (i) $f$ is linear: $f(a\phi_1 + b\phi_2) = af(\phi_1) + bf(\phi_2)$ for each $a, b \in \mathbb{R}$, $\phi_1, \phi_2 \in \mathcal{D}$, and
  (ii) $f$ is continuous with respect to its argument.
- As a linear functional, it is sometimes customary to denote $f(\phi)$ by $(f, \phi)$, in which we will use the latter in the remaining parts of this supplement.
- Every locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ defines a distribution $\tilde{f} \in \mathcal{D}'$ by
$$(\tilde{f}, \phi) = \int_{\mathbb{R}^n} f(x)\phi(x)\mathrm{d}x.$$
  These distributions are known as **regular distributions**.
- Not every distribution is given by a locally integrable function. Example: $f = \delta_0$ (the “delta function” at $0$). These distributions are known as **singular distributions**.

Given a PDE, more often than not, we are unable to define differentiability in the classical sense (i.e if the solution is only continuous, we might not be able to “differentiate” it). Hence, this motivates the notion of a **weak solution**, and makes use of the fact that a weak solution is in fact viewed as a regular distribution! The idea of a weak solution is that **if sufficiently regularity on the solution is present, then this means that the solution solves the PDE classically**. We will see an example of this for a conservation law (which will be important for the contents covered next week) as follows.

**Example 24.** Consider the Cauchy problem for a 1D scalar conservation law below:

$$\begin{cases} u_t + (f(u))_x = 0 & \text{for } (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases} \tag{169}$$

We have seen in Discussion 3 that it is possible for characteristics to collide, causing a jump discontinuity in the solution $u$ across $x$ for some time $t > 0$. However, if this is the "worst case scenario" for a solution, this implies that the solution **remains bounded** for all $x \in \mathbb{R}$ and $t > 0$. Henceforth, we can expect a solution $u(x,t) \in L^\infty(\mathbb{R} \times [0,\infty))$ to solve the above conservation law in the weak sense.

Derive the notion of a weak solution $u(x,t) \in L^\infty(\mathbb{R} \times [0,\infty))$ to the conservation law (169) above.

Suggested Solution: Let $\varphi \in C_c^\infty(\mathbb{R} \times [0,\infty))$ be a test function. Multiplying the PDE by $\varphi$ and integrating over $(x,t) \in \mathbb{R} \times [0,\infty)$ yields

$$0 = \int_0^\infty \int_{-\infty}^\infty (u_t + (f(u))_x)\varphi(x,t)\mathrm{d}x\mathrm{d}t. \tag{170}$$

Integrating by parts (in time for the first term, and in space for the second term), for a sufficiently regular solution $u$, we thus have[27]

$$0 = -\int_0^\infty \int_{-\infty}^\infty u(x,t)\varphi_t(x,t)\mathrm{d}x\mathrm{d}t - \int_{-\infty}^\infty u\varphi(x,0)\mathrm{d}x - \int_0^\infty \int_{-\infty}^\infty f(u)\varphi_x(x,t)\mathrm{d}x\mathrm{d}t. \tag{171}$$

Since $u(x,0) = u_0(x)$, we thus have

$$-\int_0^\infty \int_{-\infty}^\infty u(x,t)\varphi_t(x,t)\mathrm{d}x\mathrm{d}t - \int_{-\infty}^\infty u_0(x)\varphi(x,0)\mathrm{d}x - \int_0^\infty \int_{-\infty}^\infty f(u)\varphi_x(x,t)\mathrm{d}x\mathrm{d}t = 0 \tag{172}$$

Hence, we say that a weak solution $u(x,t) \in L^\infty(\mathbb{R} \times [0,\infty))$ if for each test function $\varphi \in C_c^\infty(\mathbb{R} \times [0,\infty))$, we have that (172) holds.

Remark: Here some remarks with regards to weak solutions to the scalar conservation laws and weak solutions in general:

- Using the notion of a weak solution, we can define a relation satisfied by the value of $u$ on the left and the value of $u$ on the right of a discontinuity, also known as the **Rankine-Hugoniot** jump condition. We will talk more about this next week.
- In general, for higher dimensions and systems of conservation laws, it is possible for solutions to have a "delta" jump, that is, $u(x,t) \approx \delta(x - s(t))$ along some "jump curve" $s(t)$. In that case, we cannot use the notion of a weak solution, and we have to resort to dealing with the conservation law in the sense of distributions. In fact, the solution is a "singular distribution", or in the literature, is known as a **singular shock** or a **$\delta$-shock**.
- For general elliptic equations, the same can be done to define the appropriate notion of a weak solution. In that case, it is customary to **NOT** shift all derivatives to the test functions as this would imply that we are looking at a really weak notion of a weak solution. For more information, see Shearer and Levy Chapter 11.2 or Evans Chapter 6.

[27] Here, we use the fact that $\varphi$ has compact support, though this does not mean that $\varphi$ vanishes at $t = 0$.

Qual problems for additional practice:

**Exercise 32.** (Spring 16, Problem 3, Elliptic.)

In this question, you will construct a Green's function solution to Poisson's equation in the domain, $[0,1]\times\mathbb{R}^2$, that is, solve:

$$\Delta u = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$$

in the domain $0 \le x \le 1$, $-\infty < y, z < \infty$. You may assume that $u \to 0$ as $|y| \to \pm\infty$ or $|z| \to \pm\infty$, and the boundary conditions on the walls $x = 0, x = 1$ are:

$$u(0,y,z) = \frac{\partial u}{\partial x}|_{(1,y,z)} = 0.$$

Seek a solution of the form:

$$u(x,y,z) = \frac{1}{(2\pi)^2}\int\int e^{i(ly+mz)}\hat{u}(x,l,m)\mathrm{d}l\mathrm{d}m \tag{173}$$

and find the function $\hat{u}$. You do not need to evaluate the integral (173).

**Exercise 33.** (Fall 16, Problem 1, Elliptic.)

Show that $u(r) = -\frac{1}{4\pi}r$, where $x \in \mathbb{R}^3$ and $r = |x|$, satisfies $\Delta u = \delta(x)$ in the sense of distribution. In other words, show that

$$\int u(x)\Delta\phi(x)\mathrm{d}x = \phi(0)$$

for any smooth $\phi$ with compact support.

**Exercise 34.** (Spring 22, Problem 4, Parabolic.)

Let $u(x,t;y)$, with $x, y, t > 0$, be a Green's function solution of the partial differential equation

$$\frac{\partial u}{\partial t} = u + \frac{\partial^2 u}{\partial x^2},$$

with boundary conditions $u(0,t;y) = 0, u(\infty,t;y) = 0$ and initial condition $u(x,0;y) = \delta(x-y)$. By explicitly deriving a formula for the solution $u$, show that it satisfies the reciprocity property $u(x,t;y) = u(y,t;x)$.

*[Note: If you make use of the fundamental solution of the heat equation in your solution, then you should state its formula. However, you do not need to prove it.]*

# 10 Discussion 10

<u>Scalar Conservation Laws.</u>
(Shearer and Levy Chapters 13.1 - 13.2, Evans Chapter 3.4.)

For a scalar conservation law of the form

$$u_t + (f(u))_x = 0,$$

the notion of a weak solution is covered in the previous discussion for a initial value problem. If $u$ is differentiable in $x$, we have

$$u_t + f'(u)u_x = 0,$$

which implies that along characteristics $(x(t), t)$, we have

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f'(u(x(t)))$$

corresponding to the speed of characteristics. Since $\frac{\mathrm{d}u(t)}{\mathrm{d}t} = 0$ by the PDE, $u$ is constant along characteristics, so we have

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f'(u(x(0))).$$

This implies that we are looking at characteristics of constant speed, with speed depending on the point along the initial data (or boundary data) for which we are starting the characteristic line on. The above formalism then supports solving the conservation law directly forward in time (as compared to the usual method that we've done in the first few weeks by finding $x(0)$ such that characteristics hit $(x(t), t) = (x, t)$ for a given $(x, t)$ in spacetime).

Hence, as we solve the initial value forward in time, one usually encounters the following two scenarios:

- Colliding characteristics. This yields a solution that is possibly discontinuous in $x$. Hence, for any time $t$ after the time in which these characteristics collide, we let $x = \gamma(t)$ denote the curve of discontinuity. The **Rankine-Hugoniot** jump condition states that the speed[28] of the curve $s$ at time $t$ is given by

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{174}$$

  Here, $u_+$ and $u_-$ refer to the value of $u$ to the left and right of the curve of discontinuity respectively. Such a curve of discontinuity is a **shock** if it satisfies the **entropy condition** below:

$$f'(u_-) > s > f'(u_+). \tag{175}$$

  Physically, this condition says that characteristic curves "collide" onto the shock curve on both sides, since

$$\underbrace{f'(u_-)}_{\text{Speed of characteristics from the left}} > \underbrace{s}_{\text{Speed of shock (curve)}} > \underbrace{f'(u_+)}_{\text{Speed of characteristics from the right}}.$$

  These properties can be derived by means of a vanishing-viscosity analysis.[29]

- Lack of solutions in certain region in spacetime. To "fill" up the solution in this regime, we do so by connecting the left and right states[30] with a smooth solution. This solution is obtained by looking for a self-similar (or scale-invariant) solution to the conservation law. By searching for solutions of the form

$$u(x, t) = \hat{u}(x/t),$$

  we can plug this into the conservation law to see that the corresponding function for $\hat{u}$ is given by

$$\hat{u}(x/t) = (f')^{-1}(x/t).$$

  For the inverse of $f'$ to exist, we assume that the flux function $f$ is strictly convex, and hence $f'$ is strictly increasing. Since the conservation law is invariant under translations in space and time (ie $x \mapsto x + x_0$ and

[28] In this chapter, we will use the word 'speed' and 'velocity' interchangeably.
[29] This is equivalent to looking for a travelling wave solution connecting the left and right state as in Shearer and Levy Chapter 12.1.
[30] In this chapter, we will use the word 'state' and 'the value of $u$' interchangeably.

$t \mapsto t + t_0$ for any $x_0, t_0$), then we have $u(x,t) = \hat{u}(x/t) = (f')^{-1}\left(\frac{x-x_0}{t-t_0}\right)$ for appropriate choice of $x_0$ and $t_0$.

For more details of the derivation, refer to Evans Chapter 3.4, page 155. Note that this is known as a **rarefaction fan** connecting the left to the right state.

We say that we have a **weak entropy solution** if we have a weak solution such that

1. Curves of discontinuities are shocks (ie satisfying the entropy condition (175)), and
2. The speed of shocks is given by (174) (satisfies the Rankine-Hugoniot jump conditions).

Examples of how to solve the conservation law for a weak entropy solution are given below.

**Example 25.** (Fall 18, Problem 7(b), Hyperbolic.) Consider the nonlinear equation

$$u_t + (u^3)_x = 0$$

for viscous flow down an inclined plane. Solve the initial-boundary value problem in the domain $x > 0, t > 0$ with boundary conditions $u(x,0) = 0$ and $u(0,t) = 1$. Here, $x = 0$ corresponds to a gate that releases fluid with height $u(0,t)$. Draw a characteristic diagram for this problem.

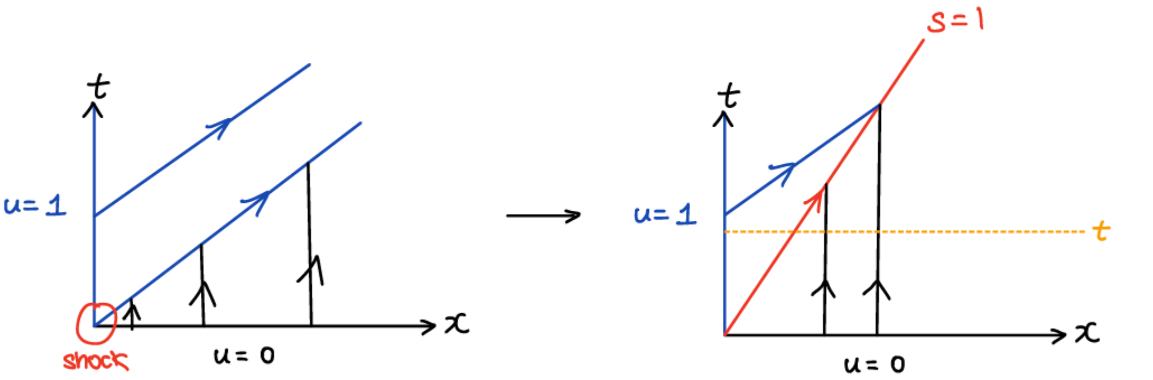
Suggested Solutions: By the PDE, we have

$$u_t + 3u^2 u_x = 0.$$

This implies that the speed of characteristics is given by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = 3u(t)^2.$$

To solve the nonlinear initial-boundary value problem, we draw the characteristic diagram for small time as follows:




Observe that the characteristics collide for points close to the origin $(x,t) = (0,0)$. To determine the speed of the shock (with $u_- = 1$ and $u_+ = 0$ to represent the value of $u$ on the left and right of the "shock"), we use the Rankine-Hugoniot jump condition to obtain

$$s = \frac{u_-^3 - u_+^3}{u_- - u_+} = u_-^2 + u_- u_+ + u_+^2 = 1^2 + 1 \cdot 0 + 0^2 = 1.$$

Since the shock curve must originate from the origin (as the characteristics close to that point starts to collide), we have the following solution for all $t > 0$:

$$u(x,t) = \begin{cases} 1 & \text{for } x < t, \\ 0 & \text{for } x > t. \end{cases}$$

Here, we note that the shock curve $\gamma(t) = t$ for all $t > 0$ since it has a constant speed of $1$ with $\gamma(0) = 0$. $\square$

**Example 26.** (Fall 23, Problem 7, Hyperbolic.) A ferry of length $L$ pulls up to a terminal with cars filled to capacity at the maximum density $\rho = 1$. When the ferry docks, the gate opens and the cars leave the ferry. Assume that the car density $\rho$ evolves according to the evolution equation

$$\rho_t + (v\rho)_x = 0 \tag{176}$$

where

$$v = \begin{cases} 1-\rho & \text{for } 0 < \rho < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{177}$$

At the "maximum packing density" $\rho = 1$, we assume that cars are parked so close together that they cannot move. The initial condition is then $\rho = 1$ for $-L < x < 0$ and $\rho = 0$ otherwise.

(i) Solve the Riemann problem for equation (176) with initial conditions $\rho = 1$ for $x < 0$ and $\rho = 0$ for $x > 0$ and $v$ defined by equation (177).
*[Hint: You may use the fact that for $F(\rho) = \rho(1-\rho)$, we have $(F')^{-1}(x/t) = 0.5(1 - x/t)$.]*

(ii) Use your solution in (i) to solve for $\rho(x,t)$ on the interval $-L < x < \infty$ for all $t > 0$ with the ferry initial condition $\rho = 1$ for $-L < x < 0$ and $\rho = 0$ otherwise.

The solution $\rho(x,t)$ should be broken down into two parts, (a) the solution during which some of the cars are densely packed and cannot move; and (b) the solution after all of the cars have started moving.

(iii) If you are in a car at the back of the ferry (i.e., at $x = L$ at $t = 0$), what is the trajectory of your car for all $t > 0$?
*[You can break the trajectory down into the time it takes for the cars in front of you to start moving (during which time your car cannot move) and then the trajectory that you follow once your car starts moving. Note that the car will travel with speed $v(x,t)$, rather than the characteristics speed that is associated with the conservation law.]*

Suggested Solutions:

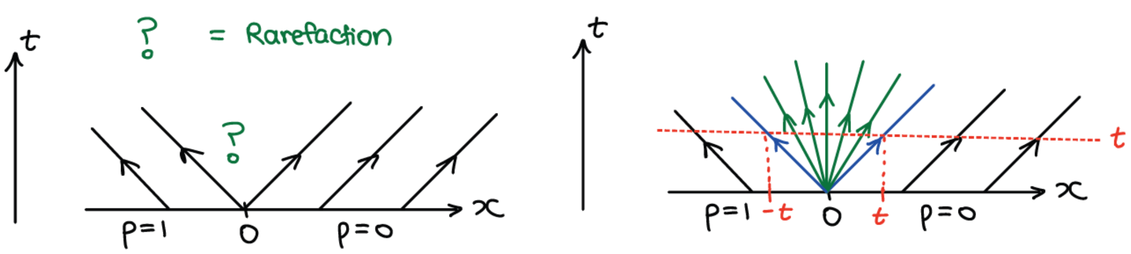
(i) By (176), we have

$$\rho_t + (1-2\rho)\rho_x = 0$$

for $\rho \in (0,1)$. This implies that the speed of characteristics is given by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = 1 - 2\rho(t)$$

along characteristics, with density constant along characteristics (since $\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = 0$).
To solve the Riemann problem, we draw the characteristic diagram for small time as follows:



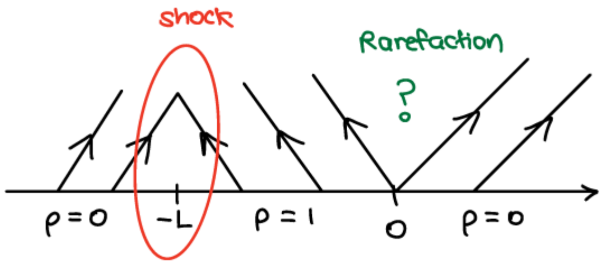

For $x < 0$, since $\rho = 1$, then $\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - 2 = -1$, corresponding to characteristic lines with velocity $-1$. Similarly, for $x > 0$, since $\rho = 0$, then $\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - 0 = 1$, corresponding to characteristic lines with velocity $1$.
From the characteristics diagram, this leaves a "gap" in which we will have to fill up the solution in it. The correct choice here would be a rarefaction fan, with formula

$$\rho(x,t) = (F')^{-1}(x/t) = \frac{1}{2}\left(1 - \frac{x}{t}\right).$$

For each time slice $t$ (indicated with the red dotted lines), we see that the rarefaction fan connects the last characteristic line emanating from $0$ to the left, to the last characteristic line emanating from $0$ to the right. Since former has a velocity of $-1$, the left end-point at time $t$ is given by $-t$, while using a symmetric argument, the right end-point at time $t$ is given by $t$. Henceforth, the solution is given by

$$\rho(x,t) = \begin{cases} 1 & \text{for } x < -t, \\ (F')^{-1}\left(\frac{x}{t}\right) = \frac{1}{2}\left(1 - \frac{x}{t}\right) & \text{for } -t < x < t, \\ 0 & \text{for } x > t. \end{cases}$$

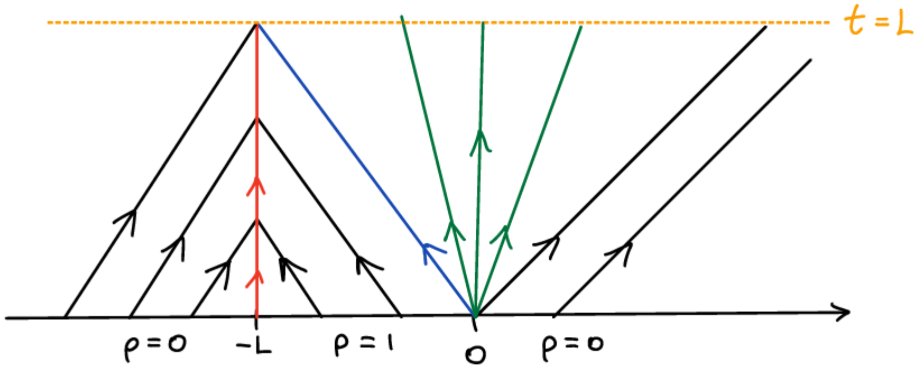
(ii) As per usual, we will draw the characteristic diagram for small time as follows:




Observe that at $x$ close to $0$, we have the same rarefaction fan structure as in (i). For $x$ close to $-L$, the two characteristics from the left (with velocity $1$) and from the right (with velocity $-1$) collides. By the Rankine Hugoniot jump condition, this forms a shock with speed $s$ given by (with $F(\rho) = \rho(1-\rho)$)

$$s = \frac{F(\rho_L) - F(\rho_R)}{\rho_L - \rho_R} = \frac{0 \times (1-0) - 1 \times (1-0)}{0 - 1} = 0.$$

This implies that the two collding characteristics from both sides of $x = -L$ forms a stationary shock at $x = -L$ (since the shock speed is $0$). Hence, we have the following structure for small time:




Observe that in the diagram above, the shock is always fed with characteristic lines from the left and characteristic lines from the right (from $\rho = 1$). However, observe that past a certain time, we have the final characteristic line from the $\rho = 1$ part of the initial data (indicated in blue) colliding with the shock. After this time, the information feeding the shock from the right comes from a rarefaction fan with a lower value of $\rho$.[31] To compute this time, it suffices to compute the point of intersection between the shock curve $x = -L$ and the characteristic line $x = -t$. Hence, we observe that the intersection occurs at $t = L$.

Next, we would like to compute what happens after time $t = L$. Here, we observe that the shock is fed with the same information from the left with characteristic lines emanating from $\rho = 0$. On the right, on each point of the shock curve, we find the corresponding characteristic line emanating from the rarefaction fan as follows.

Assume that at time $t > L$, we have the shock curve parameterized by $(\gamma(t), t)$, where $x = \gamma(t)$ is the location of the shock curve. At this point, to the left of the shock curve, we have $\rho = 0$ (this piece of

[31] Recall that a rarefaction fan connects two states, in this case, 1 to 0, in a smooth manner, with each "line" corresponding to a different value $\rho$. In this case, it corresponds to a lower value of $\rho$, and thus resembles a characteristic line with a higher velocity (since the velocity is given by $1 - 2\rho$).

information is carried by characteristic lines from $\rho = 0$ along the initial data from the left). On the other hand, on the right of the shock curve, we use the formula $\rho(\gamma(t)^+, t) = \frac{1}{2}\left(1 - \frac{\gamma(t)}{t}\right)$ from (i). By the Rankine-Hugoniot jump condition, we have that the shock speed (given by $\gamma'(t)$), is given by

$$\gamma'(t) = s = \frac{F(0) - F\left(\frac{1}{2}\left(1 - \frac{\gamma(t)}{t}\right)\right)}{0 - \frac{1}{2}\left(1 - \frac{\gamma(t)}{t}\right)} = \frac{-\frac{1}{2}\left(1 - \frac{\gamma(t)}{t}\right)\frac{1}{2}\left(1 + \frac{\gamma(t)}{t}\right)}{-\frac{1}{2}\left(1 - \frac{\gamma(t)}{t}\right)} = \frac{1}{2}\left(1 + \frac{\gamma(t)}{t}\right). \tag{178}$$

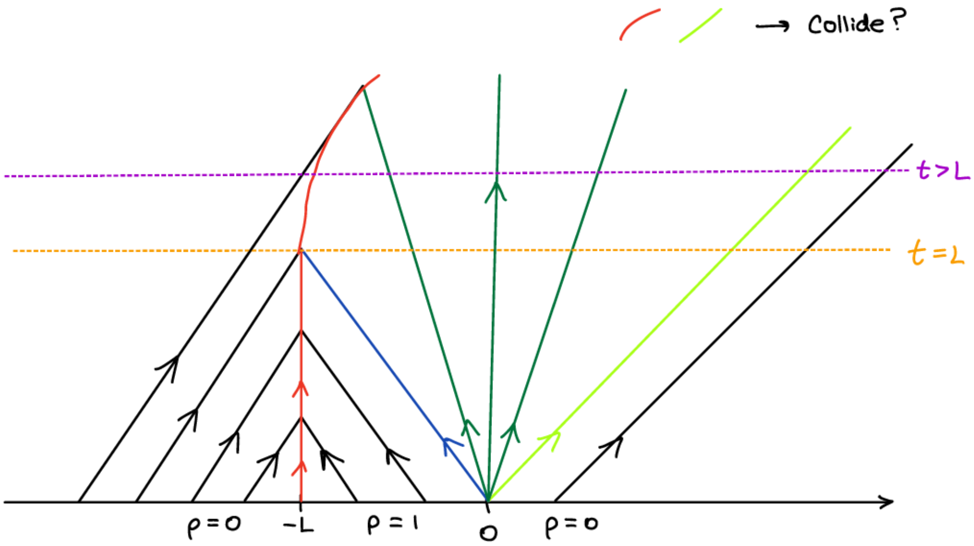
The corresponding initial data for this ODE is given by

$$\gamma(L) = -L \tag{179}$$

corresponding to the point in spacetime for which the shock curve starts to branch out from. The solution to the ODE is given by

$$\gamma(t) = -2\sqrt{Lt} + t \quad \text{ for } t > L. \tag{180}$$

Hence, we have the following characteristic diagram for $t > L$:




Note that the shock curve and the characteristic line at the right end of the rarefaction fan does not intersect, since $\gamma(t) < t$ for all $t > L$ and $x(t) = t$ is the equation of the characteristic line. Henceforth, we have the following solution for all $t > 0$: For $t \in (0, L)$, we have

$$\rho(x,t) = \begin{cases} 0 & \text{for } x < -L, \\ 1 & \text{for } -L < x < -t, \\ \frac{1}{2}\left(1 - \frac{x}{t}\right) & \text{for } -t < x < t, \\ 0 & \text{for } x > t, \end{cases}$$

and for $t > L$, we have

$$\rho(x,t) = \begin{cases} 0 & \text{for } x < \gamma(t), \\ \frac{1}{2}\left(1 - \frac{x}{t}\right) & \text{for } \gamma(t) < x < t, \\ 0 & \text{for } x > t. \end{cases}$$

(iii) Here, we assume that the velocity of the car is given by the mean velocity in the macroscopic model. Hence, along the path of a car (denoted by $p(t)$)[32] at the back of the ferry (in Lagrangian coordinates), we have

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = v(p(t), t) = 1 - \rho(p(t), t).$$

[32] We use $p(t)$ to distinguish between $x(t)$, which we have used to indicate the path of characteristic lines. As mentioned in the hint and in general, they are not necessarily equal.

Hence, to determine the trajectory of the car, we assume that we start with $p(0) = (-L)^+$, that is, we are just at the back of the bunch of cars behind the gate once the gate is open. For all $t < L$, we have $\rho(p(t), t) = 1$ and hence $\frac{\mathrm{d}p(t)}{\mathrm{d}t} = 0$ and the car is stationary. This corresponds to the cars clearing through the gate and the car at the back of the bunch would have to wait for the cars in front to first clear the gate. Hence, we have

$$p(t) = -L \quad \text{for } t \in (0, L).$$

Once the rarefaction fan starts to hit $x = -L$, the density starts to fall "below the maximum packing density". Hence, using $\rho(x,t) = \frac{1}{2}\left(1 - \frac{x}{t}\right)$, we have the following differential equation for the kinematics of the car of interest:

$$\begin{cases} \frac{\mathrm{d}p(t)}{\mathrm{d}t} = 1 - \frac{1}{2}\left(1 - \frac{p(t)}{t}\right) = \frac{1}{2}\left(1 + \frac{p(t)}{t}\right) & \text{for } t > L, \\ p(L) = -L. \end{cases}$$

Observe that this is the same exact ODE for the shock curve in (178) and (179) of (ii). Hence, the solution is identical to (180) and is given by

$$p(t) = \gamma(t) = -2\sqrt{Lt} + t \quad \text{for } t > L.$$

As explained in (ii), the shock curve will never intersect the final characteristic line on the rarefaction fan. Hence, the expression for $\rho(x,t)$ used in forming the ODE will not change, we will have the same ODE for all $t > L$. Hence, the trajectory is given by $p(t) = \gamma(t)$ for all $t > 0$. Explicitly,

$$p(t) = \begin{cases} -L & \text{for } 0 < t \leq L, \\ -2\sqrt{Lt} + t & \text{for } t > L. \end{cases}$$

$\square$

Nonconvex Scalar Conservation Laws.
(Shearer and Levy Chapters 13.3 and 13.4.)

Recall that to construct a rarefaction fan, we require the invertibility of the function $f'$, which cannot occur on its domain if $f$ is not strictly convex.

To resolve this, we resort to a more general form of the entropy condition as follows:

**Olenik (Chord) Entropy Condition.** A weak solution $u(x,t)$ is the vanishing-viscosity solution to a general scalar conservation law if all discontinuities have the property that

$$\frac{f(u)-f(u_l)}{u-u_l} \geq s \geq \frac{f(u)-f(u_r)}{u-u_r}$$

for all $u$ between $u_l$ and $u_r$, with the shock speed $s$ determined by the Rankine-Hugoniot jump condition in (174). By plugging the expression for the shock speed in, we have

$$\frac{f(u_l)-f(u)}{u_l-u} \geq \frac{f(u_l)-f(u_r)}{u_l-u_r} \geq \frac{f(u_r)-f(u)}{u_r-u} \tag{181}$$

for all $u$ between $u_l$ and $u_r$. The graphical interpretation of condition (181) is that for a given $u_l$ and our candidate choice of $u_r$, we must have that the chord joining $(u_l, f(u_l))$ and $(u_r, f(u_r))$ lie above all chords joining $(u_l, f(u_l))$ to $(u, f(u))$, and $(u_r, f(u_r))$ to $(u, f(u))$.

In general, this might not be true for a given $u_l$ and $u_r$. Hence, the work around is as follows. Without loss of generality, consider $u_l > u_r$. Since $u_r$ lies to the left of $u_r$ on the $u-$axis, we start by finding the minimum value of $u$ (closest to $u_r$) such that the "shock chord" connecting $(u_l, f(u_l))$ and $(u_*, f(u_*))$ is above all chords joining $(u_l, f(u_l))$ to $(u, f(u))$, and $(u_*, f(u_*))$ to $(u, f(u))$ for all $u \in (u_*, u_l)$ (ie over a region for which the curve $f(u)$ is convex in $u$). By (181), this corresponds to a point for which the "shock chord" is tangent to the point $(u_*, v_*)$. Given that we are unable to pick a smaller value of $u$ such that the graphical condition holds, we have to resort to "following" along the concave region to form a "rarefaction curves" (remember that we can always invert the function $f'$ when it is strictly monotone). This process is repeated to create the **upper convex envelope** of the flux function. For $u_r > u_l$, one form the **lower convex envelope** of the flux function.

For example, consider the conservation law

$$u_t + (u^3)_x = 0$$

with Riemann initial data

$$u(x,0) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x > 0. \end{cases}$$

Suppose that $u_l = 1$ and $u_r = -1$. By the "algorithm" mentioned above, we form the upper convex envelope as follows. First, we determine the point on the graph $f(u)$ for which the shock chord is tangent to the graph at that point. This implies that

$$3u^2 = f'(u) = \frac{f(u_l)-f(u)}{u_l-u} = \frac{1^3-u^3}{1-u} = 1^2+u+u^2.$$

This is to be solved for $u \neq 1$ since the Rankine-Hugoniot condition is not defined for the left and right states to be equal, and this is also an expected value given that a chord with vanishing width approaches the tangent of a graph by calculus.

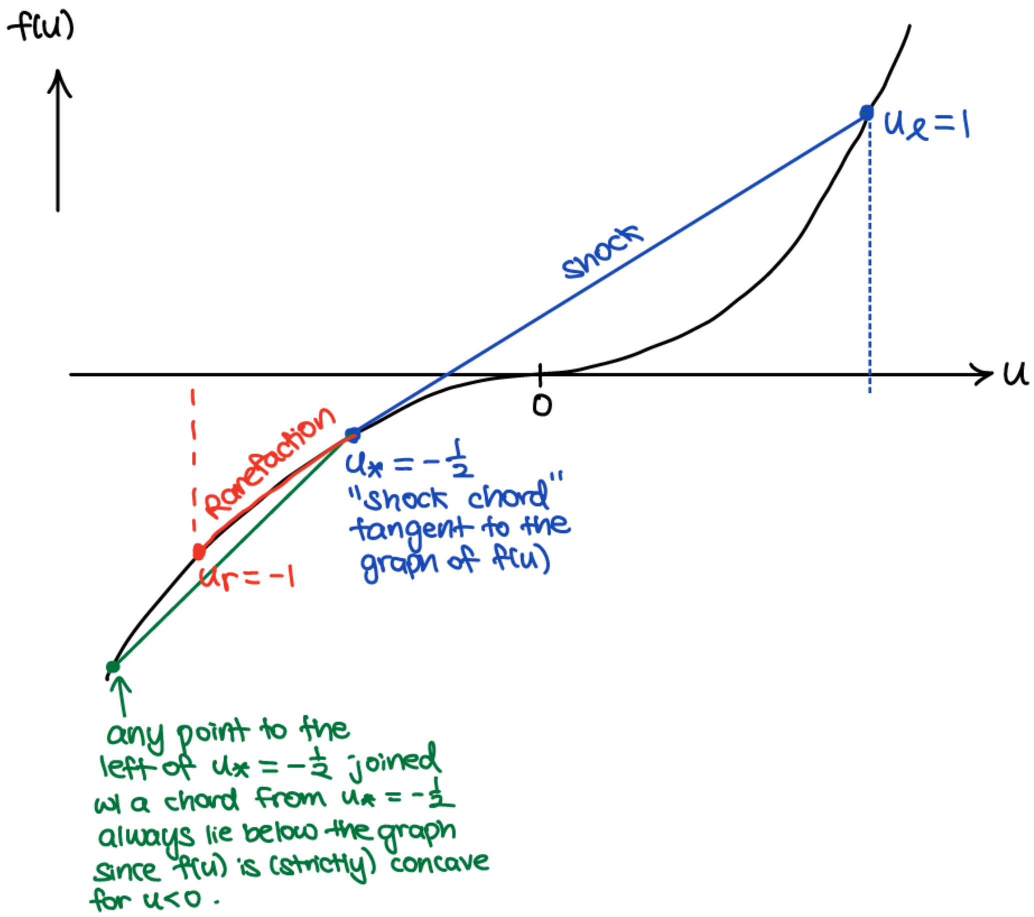
This simplifies to

$$2u^2 - u - 1 = 0,$$

with solutions $u = 1$ or $u = -\frac{1}{2}$. Since $u \neq 1$, then $u_* = -\frac{1}{2}$. Afterwards, we move "along the concave" component of the graph to reach 1.

Since the graph is concave for $u < -1/2$, the graph will always lie above any chord connecting two points on the graph. Hence, the "Olenik chord condition" can never be satisfied at any point further down with $u < -\frac{1}{2}$.

The following diagram further illustrates the aforementioned algorithm.




The Rankine-Hugoniot jump condition gives the speed of the shock connecting the state $u_l = 1$ and $u_* = -\frac{1}{2}$, given by

$$s = \frac{f(u_l) - f(u_*)}{u_l - u_*} = u_l^2 + u_l u_* + (u_*)^2 = \frac{3}{4}.$$

Thus, the solution $u(x,t)$ for $t > 0$ is given by

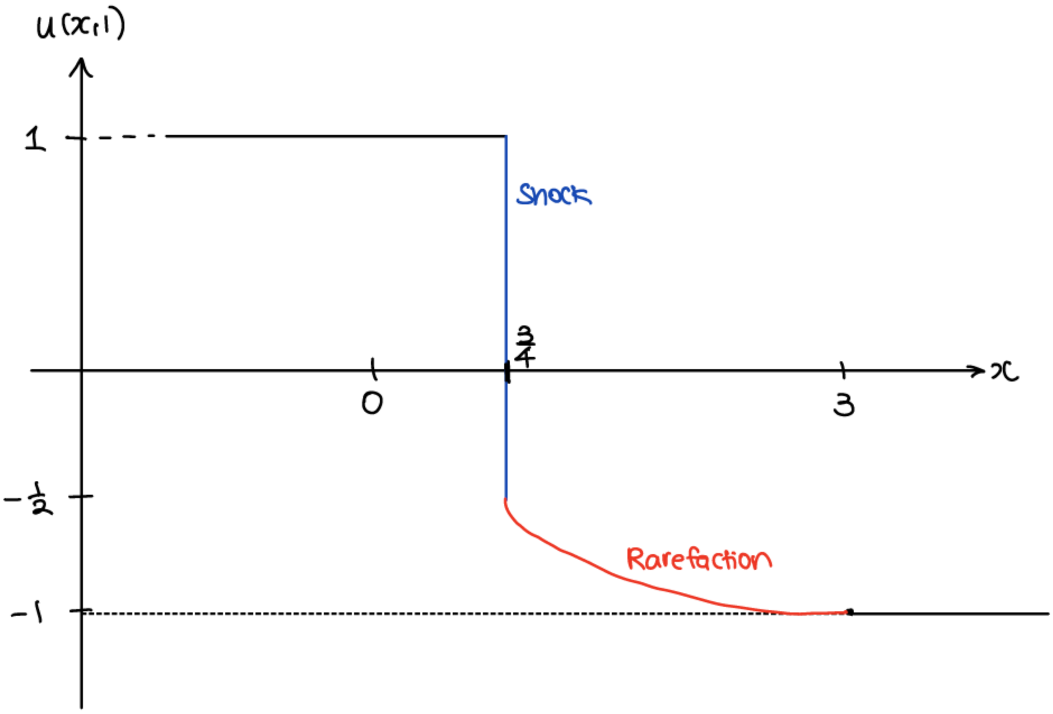
$$u(x,t) = \begin{cases} 1 & \text{for } x < \frac{3}{4}t, \\ -\sqrt{\frac{x}{3t}} & \text{for } \frac{3}{4}t < x < 3t, \\ -1 & \text{for } x > 3t. \end{cases}$$

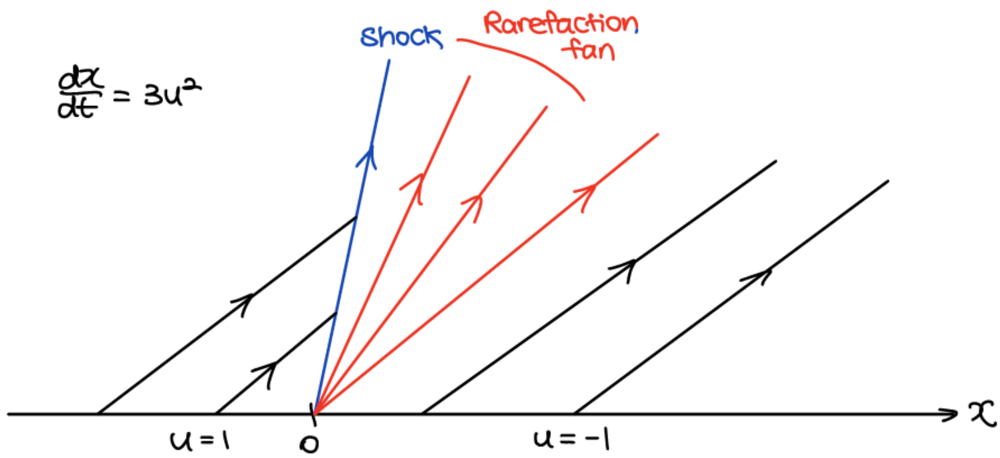
To determine the expression of the rarefaction fan, recall that this is given by $(f')^{-1}(x/t)$. The function $f'(u) = 3u^2$. For $u < 0$, the inverse function is thus given by $(f')^{-1}(u) = -\sqrt{\frac{u}{3}}$. Hence, the expression for the rarefaction solution is given by

$$u(x,t) = (f')^{-1}(x/t) = -\sqrt{\frac{x}{3t}}.$$

Since this rarefaction fan connects $-1/2$ to $-1$, then we must have $-1 \leq -\sqrt{\frac{x}{3t}} \leq -1/2$, in which one can then solve this to obtain $\frac{3}{4}t < x < 3t$.

The solution at time $t = 1$ and the corresponding characteristic diagram are shown below:




Observe from the characteristic diagram that in the absence of the rarefaction fan, it is possible to admit an **undercompressive shock**, that is, characteristics only collide onto the shock from one direction. In engineering literature, these are known as **subsonic shocks** (since the speed of the shock is slower than the speed of fluid adjacent to it). These shocks do not satisfy the Lax entropy condition (nor the Olenik entropy condition), though they do exists from physical experiments. To mathematically motivate the existence of an undercompressive shock, one has to resort to including the vanishing-“something” solution to a conservation law, where we include regularizing terms with higher order (ie $u_{xxx}$ such as in the KdV equation, or $u_{xxxx}$ representing effects of surface tension in fluids) on top of the usual viscosity term $u_{xx}$. For more information, see Shearer and Levy Chapter 13.4.

Qual problems for additional practice:

**Exercise 35.** (Spring 23, Problem 6, Hyperbolic.) Consider the PDE:

$$u_t + uu_x = -u$$

with initial condition

$$u(x,0) = 1, x < 0; \quad u(x,0) = 0, x > 0.$$

(i) Show that for a smooth solution, the PDE can be written in terms of the "characteristics" variable $x = \xi(t)$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}u(\xi(t)) = -u(\xi(t)); \quad \frac{\mathrm{d}\xi(t)}{\mathrm{d}t} = u(\xi(t)).$$

(ii) Suppose that $u(0,0)$ takes values $\alpha \in (0,1)$. Solve for the characteristics $\xi_\alpha(t)$ starting from $x = 0$ with initial value $u = \alpha$.

(iii) Using your result in (ii), solve the Riemann problem with the initial condition above. Write your answer in terms of the Eulerian variables $x$ and $t$.

**Exercise 36.** (Spring 22, Problem 8, Hyperbolic.) Consider the following hyperbolic conservation law for traffic flow:

$$u_t + (u(1-u))_x = 0,$$

where $u$ is the density of vehicles and $1-u$ is the mean speed of vehicles at density $u$. Note that the flux of vehicles is the mean speed multiplied by the mean density.

(i) Solve the Riemann problem for the case of vehicles stopped at a traffic light that turns green at time $t = 0$ :

$$u(x,0) = \begin{cases} 1, & x < 0, \\ 0, & x \geq 0. \end{cases}$$

(ii) Solve the Riemann problem for the case of congestion on a road:

$$u(x,0) = \begin{cases} 0.25, & x < 0, \\ 0.75, & x \geq 0. \end{cases}$$

(iii) In the congestion case in part (ii), suppose that you are travelling in a vehicle whose speed is the mean speed that is described above. Your vehicle's position starts at $x = -1$ at time $t = 0$.

What is your path $x(t)$ going forward in time?
*[Hint: This is not a characteristic. Use the solution to (ii) to determine the velocity of your vehicle before and after it enters the congestion region.]*

**Exercise 37.** (Fall 21, Problem 6, Hyperbolic.) Consider Burgers equation $u_t + uu_x = 0$ for $\{x, t > 0\}$ with initial condition $u(x,0) = 0$ and boundary condition $u(0,t) = 1$ for $0 < t < 1$ and $u(0,t) = 2$ for $t > 1$.

(i) Plot the characteristics coming from the $t-$axis.

(ii) What is the entropy solution for $0 < t < 1$?
*[Hint: It should be related to the Riemann problem.]*

(iii) For $t > 2$, there is a new structure emerging from the $t-$axis. What is it? When will it merge with the structure from part (ii)?

(iv) Write down the full solution to the problem for $t > 0$, satisfying the entropy condition.

**Exercise 38.** (Spring 21, Problem 5, Hyperbolic.) Consider the conservation law

$$\begin{cases} u_t + (u^3)_x = 0 & \text{in } \mathbb{R} \times (0, \infty); \\ u = g & \text{in } \mathbb{R} \times \{t = 0\}. \end{cases}$$

We refer to *entropy solutions* as integral solutions which satisfy the Lax entropy condition.

(i) Find an entropy solution when

$$g(x) = 1 \quad \text{for } 0 < x < 1, \text{ otherwise } g = 0.$$

(ii) Find an integral solution of the above problem that is not an entropy solution.

**Exercise 39.** (Spring 19, Problem 4, Hyperbolic.) Let $u$ be the entropy satisfying weak solution of

$$u_t + f(u)_x = 0, x \in \mathbb{R}, t > 0,$$
$$u(x, 0) = \begin{cases} u_a, & x < 0, \\ u_b, & x > 0, \end{cases}$$

with $f(u) = \frac{u^2}{2}$ with $u_a, u_b > 0$.

(i) Show that

$$\int_a^b u(x, t)\mathrm{d}x - \int_a^b u(x, 0)\mathrm{d}x = t(f(u_a) - f(u_b))$$

for $t < T$ for some $T > 0$.

(ii) Give an expression for $T$ with $u_a < u_b$.

(iii) Give an expression for $T$ with $u_a > u_b$.

**Exercise 40.** (Fall 19, Problem 7, Hyperbolic.) Consider the viscous Burgers equation

$$u_t + uu_x = \varepsilon u_{xx}, \quad x \in \mathbb{R},$$

with far-field boundary conditions $u(-\infty, t) = U_L$ and $u(+\infty, t) = u_R$.

(i) Derive the equation for a similarity solution in the form of a traveling wave $u(x, t) = U(x - st)$, and solve for $s$ in terms of $U_L$ and $U_R$.

(ii) Write a necessary and suffcient condition that $U_L$ and $U_R$ must satisfy for a traveling-wave solution to exist.

(iii) Show that the similarity solution has an additional scaling parameter $\varepsilon$. Assuming that $\varepsilon$ is a length scale, write the similarity solution in dimensionless variables with $\tilde{x} = \frac{x}{\varepsilon}$.

(iv) Compute the limiting solution as $\varepsilon \to 0$.

**Exercise 41.** (Spring 18, Problem 5, Hyperbolic.) Consider entropy solutions $u(x,t) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ of the flux-conservation equation:

$$u_t + (f(u))_x = 0$$

with initial condition

$$u(x,0) = \begin{cases} x, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and flux function $f(u) = \frac{u^3}{3}$.

(i) Derive the Rankine-Hugoniot conditon for propagation of discontinuous solutions of this PDE.

(ii) Find the long time solution of the PDE. You may assume that $u \geq 0$, so $f(u)$ is convex, and also that at long times, the solution can be broken into three parts:

$$u(x,t) = \begin{cases} 0, & \text{if } x < 0, \\ t^\alpha g\left(\frac{x}{t^\beta}\right) & \text{if } 0 < x < h(t), \\ 0, & \text{if } x > h(t), \end{cases}$$

for some exponents $\alpha$ and $\beta$, and positive functions $g$ and $h$, all of which you should determine.

# 11 Finals Revision (with Suggested Solutions.)

Scope: ((Lecture ∪ Discussion ∪ Homework) \ (Optional Discussion Materials)) ⊂ Qual Materials.

**Exercise 42.** (Fall 18, Problem 5, Modified.) Consider the following initial-boundary value problem for $u = u(x,t)$ in the domain $\{(x,t) : x > t, t > 0\}$ :

$$\begin{cases} u_t - u_{xx} + u_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = x^2 & \text{on } \mathbb{R}^+ \times \{t = 0\}, \\ u_x(x,t) = 0 & \text{on } \{(x,t) : x = t, t > 0\}, \end{cases}$$

Find an explicit solution to this problem.

Suggested Solutions: Consider the substitution:

$$\begin{cases} y &= x - t, \\ s &= t. \end{cases} \tag{182}$$

One can then show that

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial y}{\partial t}\frac{\partial}{\partial y} + \frac{\partial s}{\partial t}\frac{\partial}{\partial s} = -\frac{\partial}{\partial y} + \frac{\partial}{\partial s}, \\ \frac{\partial}{\partial x} &= \frac{\partial y}{\partial x}\frac{\partial}{\partial y} + \frac{\partial s}{\partial x}\frac{\partial}{\partial s} = \frac{\partial}{\partial y}. \end{aligned} \tag{183}$$

By writing the PDE in the new coordinates $(y,s)$, we have

$$\begin{cases} u_s - u_{yy} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(y,0) = y^2 & \text{on } \mathbb{R}^+ \times \{s = 0\}, \\ u_y(0,s) = 0 & \text{on } \{(y,s) : y = 0, s > 0\}, \end{cases}$$

Since $u_y(0,t)$, we consider the even extension of the initial data across $y = 0$, denoted by $\hat{u}$, to obtain

$$\begin{cases} \hat{u}_s - \hat{u}_{yy} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ \hat{u}(y,0) = y^2 & \text{on } \mathbb{R} \times \{s = 0\}. \end{cases}$$

Here we have used the fact that the even extension of $y^2$ remains unchanged (since its an even function in $y$). By utilizing the fundamental solution to the heat equation, we have

$$\begin{aligned} \hat{u}(y,s) &= \int_{-\infty}^{\infty} \Phi(y-\xi,s)\hat{u}(\xi,0)\mathrm{d}\xi \\ &= \frac{1}{\sqrt{4\pi s}}\int_{-\infty}^{\infty} e^{-\frac{|y-\xi|^2}{4s}} y^2 \mathrm{d}\xi \\ &= 2s + y^2. \end{aligned}$$

Since the restriction of the solution to any $y > 0$ and $s > 0$ is equals to $u(y,s)$, we then have

$$u(y,s) = 2s + y^2, \quad s > 0, y > 0.$$

Plugging the substitution back in, we have

$$u(x,t) = 2t + (x-t)^2 \quad \text{for } x > t, t > 0.$$

**Exercise 43.** (Fall 21, Problem 5 (i).) A function $f(x,t)$ satisfies the nonlinear PDE

$$\Delta u - u^3 = 0$$

on a bounded, open domain $\Omega \subset \mathbb{R}^d$ with boundary conditions $u = g(x)$ on $\partial\Omega$. Assume that $u \in C^2(\Omega) \cup C(\overline{\Omega})$, and that $g(x) > 0$ at some $x \in \partial\Omega$.

Show that $u(x) < \max_{X \in \partial\Omega}\{g(X)\}$ for all $x \in \Omega$.

Suggested Solutions:

Assume for a contradiction that there exists some $x_0 \in \Omega$ such that

$$u(x_0) \geq \max_{X \in \partial\Omega}(g(X)). \tag{184}$$

Since there is some point on $y \in \partial\Omega$ such that $g(y) > 0$, then we must have

$$u(x_0) \geq \max_{X \in \partial\Omega} g(X) \geq g(y) > 0.$$

Note that since $\Omega$ is an open and bounded subset of $\mathbb{R}^d$, then $\overline{\Omega}$ is compact and the maximum of $u \in C(\overline{\Omega})$ thus exists in $\overline{\Omega}$.

If the maximum is attained only on $\partial\Omega$, then we must have $u(x) < \max_{X \in \partial\Omega}(g(X))$ for each $x \in \Omega$, hence $u(x_0) < \max_{X \in \partial\Omega}(g(X))$, contradicting (184).

Hence, the maximum must be attained in $\Omega$, the interior of the set $\overline{\Omega}$. Let $x_1$ be the one of these maximum points. Then, we have

$$u(x_1) \geq u(x_0) > 0. \tag{185}$$

Furthermore, as a local maximum for $u \in C^2(\Omega)$, we have

$$\Delta u(x_1) \geq 0. \tag{186}$$

By the PDE, we have

$$\underbrace{\underbrace{\Delta u(x_1)}_{\leq 0} - \underbrace{(u(x_1))^3}_{>0}}_{<0} = 0, \tag{187}$$

a contradiction. Thus, we have

$$u(x) < \max_{X \in \partial\Omega}\{g(X)\}$$

as required.

**Exercise 44.** (Spring 23, Problem 4.) Consider the nonlinear PDE

$$\begin{cases} u_t = \Delta u - u^3 & x \in D, \quad 0 < t < T, \\ u(x,0) = f(x) & x \in D, \\ u(x,t) = g(x,t) & (x,t) \in \partial D \times [0,T], \end{cases}$$

where $D \subset \mathbb{R}^n$ is a bounded and open set and $f$ and $g$ are continuous functions. You may assume that solutions exist and are $C^{2,1}(D \times (0,T)) \cap C(\overline{D} \times [0,T])$. Show that the solutions of the PDE are unique.

Suggested Solutions: Let $u$ and $v$ both be solutions to the PDE above. Let $w := u - v$. Then, we observe that $w$ satisfies

$$\begin{cases} w_t = \Delta w - (u^3 - v^3) & x \in D, \quad 0 < t < T, \\ w(x,0) = 0 & x \in D, \\ w(x,t) = 0 & (x,t) \in \partial D \times [0,T]. \end{cases}$$

Since $u^3 - v^3 = (u-v)(u^2 + uv + v^2) = w(u^2 + uv + u^2)$, we can simplify this to

$$\begin{cases} w_t = \Delta w - w(u^2 + uv + v^2) & x \in D, \quad 0 < t < T, \\ w(x,0) = 0 & x \in D, \\ w(x,t) = 0 & (x,t) \in \partial D \times [0,T]. \end{cases}$$

Observe that since $|uv| \le \frac{1}{2}(u^2 + v^2)$, then

$$u^2 + uv + v^2 \ge u^2 - |uv| + v^2 \ge \frac{1}{2}(u^2 + v^2) \ge 0. \tag{188}$$

Next, suppose that we have $w(x,t) \le 0$ for all $(x,t) \in \overline{D} \times [0,T]$. By reversing the role of $u$ and $v$, we can define $\tilde{w} := v - u$, and show that it solves the same exact PDE (with $w$ replaced by $\tilde{w}$). Hence, we can apply this to obtain $-w(x,t) \le 0$, and hence $w(x,t) \ge 0$ for all $(x,t) \in \overline{D} \times [0,T]$. This in turn implies that $w \equiv 0$, and hence $u \equiv v$.

To show that $w(x,t) \le 0$ for all $(x,t) \in \overline{D} \times [0,T]$, suppose for a contradiction that there exists some $(x_0,t_0) \in \overline{D} \times [0,T]$ such that

$$w(x_0,t_0) > 0.$$

Since $w$ on $\partial D \times [0,T]$ and $D \times \{t = 0\}$ is zero, then $(x_0,t_0) \in D \times (0,T]$ (in the interior of the parabolic boundary, or at the interior of the time slice $t_0 = T$).

Observe that since $u^2 + uv + v^2 \ge 0$ (and this inequality is not strict), at a local maximum in which $w(x_1,t_1) > 0$, we will not be able to use this to conclude as:

$$\underbrace{w_t}_{\ge 0} = \underbrace{\Delta w}_{\le 0} \underbrace{- \underbrace{w}_{>0} \underbrace{(u^2 + uv + v^2)}_{\ge 0}}_{\le 0}.$$

This necessitates a perturbation argument. Consider the perturbation $w_\varepsilon := w + \varepsilon e^{-t}$ for some $\varepsilon > 0$. This implies that

$$w_t = (w_\varepsilon)_t + \varepsilon e^{-t}$$

and

$$\Delta(w) = \Delta(w_\varepsilon).$$

The PDE then reduces to

$$(w_\varepsilon)_t + \varepsilon e^{-t} = \Delta(w_\varepsilon) - (w_\varepsilon + \varepsilon e^{-t})(u^2 + uv + v^2).$$

Let $(x_1,t_1)$ be the point in $\overline{D} \times [0,T]$ for which the maximum of $w_\varepsilon$ is attained. Consider the following cases:

- If $w_\varepsilon(x_1,t_1) + \varepsilon e^{-t_1} \ge 0$ and $x_1 \in D$, then $\Delta(w_\varepsilon)(x_1,t_1) \le 0$ and $(w_\varepsilon)_t \ge 0$.[33] By the PDE, we have

$$\underbrace{(w_\varepsilon)_t(x_1,t_1)}_{\ge 0} + \varepsilon \underbrace{e^{-t_1}}_{>0} = \underbrace{\Delta(w_\varepsilon)}_{\ge 0} - \underbrace{(w_\varepsilon + \varepsilon e^{-t})}_{\ge 0} \underbrace{(u^2 + uv + v^2)}_{\ge 0},$$

[33] Recall that this is equals to 0 if $t_0 < T$, and $\ge 0$ if $t_0 = T$.

a contradiction.

- $x_1 \in \partial D$, then $w_\varepsilon(x,t) \le w_\varepsilon(x_1,t_1) = w(x_1,t_1) + \varepsilon e^{-t_1} = \varepsilon e^{-t_1}$ for each $(x,t) \in D \times [0,T]$ since the maximum is attained at $(x_1,t_1)$. This then implies that

$$w(x,t) = w_\varepsilon(x,t) - \varepsilon e^t \le \varepsilon(e^{t_1} - e^t) \le \varepsilon e^{t_1}.$$

  Since this is true for all $\varepsilon > 0$, sending $\varepsilon \to 0^+$ yields

$$w(x,t) \le 0,$$

  contradicting our initial assumption.

- $w_\varepsilon(x_1,t_1) + \varepsilon e^{-t_1} < 0$. Since the maximum of $w_\varepsilon(x,t)$ is attained at $(x_1,t_1)$, this implies that for all $(x,t) \in D \times [0,T]$, we have

$$w(x,t) = w_\varepsilon(x,t) - \varepsilon e^t \le w_\varepsilon(x_1,t_1) - \varepsilon e^t \le -\varepsilon e^{-t_1} - \varepsilon e^t \le 0.$$

  Hence, (without sending $\varepsilon$ to $0$), we obtain a contradiction with our initial assumption.

In summary, our initial assumption is false, and thus $w(x,t) \le 0$ for all $(x,t) \in \overline{D} \times [0,T]$.

Remark: Note that this could have been done by arguing that if $w > 0$ at $(x_0,t_0)$, then $u > v$ and hence $u^2 + uv + v^2 > 0$. The purpose of doing it this way is to illustrates a perturbative argument.

**Exercise 45.** (Spring 20, Problem 6.) If $U$ is the $n-$cube $\{x \in \mathbb{R}^n : -1 < x_i < 1 \quad \text{for each } i = 1, 2, \cdots, n\}$, show that the $C^2(U) \cap C(\overline{U})$ solutions to the equation

$$-\Delta u = -x \cdot \nabla u + 1 \tag{189}$$

depend continuously on the boundary data.

Specifically, show that if $\partial U$ is the boundary data of the cube, and $u_1$ and $u_2$ are solutions with $u_i(x)|_{\partial U} = g_i(x)$, then for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\|g_1 - g_2\|_{L^\infty} < \delta$ on $\partial U$, then $\|u_1 - u_2\|_{L^\infty} < \varepsilon$ in $\overline{U}$.[a]

[a]Note: $\|f\|_{L^\infty(U)} = \sup_{x \in U} |f(x)|$ for any $f \in L^\infty(U)$.

Suggested Solutions: Consider $u_i$ be the solutions to (189). Let $w = u_1 - u_2$. We observe that $w$ solves the following elliptic PDE:

$$\begin{cases} -\Delta w = -x \cdot \nabla w & \text{for } x \in U, \\ w(x)|_{\partial U} = g_1 - g_2. \end{cases} \tag{190}$$

Note: We would like to have the weak maximum principle hold for (190) to conclude. Observe that at a maximum for $w$, we have the following sign check

$$\underbrace{-\Delta w}_{\geq 0} = -x \cdot \underbrace{\nabla w}_{0},$$

which is almost a correct solution. This necessitates a perturbative argument.

Hence, we consider the perturbed $w_\varepsilon(x,t) := w(x,t) + \varepsilon\|x\|^2$. We then compute the derivatives as follows:

- $\nabla(w_\varepsilon) = \nabla w + 2\varepsilon x$, and
- $\Delta(w_\varepsilon) = \Delta w + 2n\varepsilon$.

This converts the elliptic PDE in (190) to

$$\begin{aligned} -\Delta w_\varepsilon + 2n\varepsilon &= -x \cdot (\nabla w_\varepsilon - 2\varepsilon x) \\ -\Delta w_\varepsilon &= -x \cdot \nabla w_\varepsilon - 2\varepsilon(\|x\|^2 - n). \end{aligned} \tag{191}$$

Let $x_*$ be point in $\overline{U}$ that maximizes $w_\varepsilon$ (which exists by Extreme Value Theorem). Suppose that $x_* \in U$. Hence, we have $\nabla w_\varepsilon = 0$ and $\Delta w_\varepsilon \leq 0$ at $x_*$. This implies that we have the following sign check:

$$\underbrace{-\Delta w_\varepsilon}_{\geq 0} = -x \cdot \underbrace{\nabla w_\varepsilon}_{0} \underbrace{-2\varepsilon(\|x\|^2 - n)}_{<0},$$

a contradiction. The last inequality can be observed from the fact that since $x$ is in the $n-$cube, we have each $|x_i| < 1$, $|x_i|^2 < 1$, and hence $\|x\|^2 = \sum_{i=1}^n |x_i|^2 < n$. Hence, we have the weak maximum principle for $w_\varepsilon$, that is,

$$\max_U w_\varepsilon \leq \max_{\partial U} w_\varepsilon.$$

In other words, the maximum is attained at the boundary of the set $U$.

Hence, for each $x \in U$, we have

$$w(x) = w_\varepsilon - \varepsilon\|x\|^2 \leq \max_U w_\varepsilon + 0 \leq \max_{\partial U} w_\varepsilon = \max_{\partial U}(w + \varepsilon\|x\|^2) \leq \max_{\partial U} w + \varepsilon \max_{\partial U} \|x\|^2 \leq \|g_1 - g_2\|_{L^\infty} + \varepsilon. \tag{192}$$

This then implies that

$$\|u_1 - u_2\|_{L^\infty} = \|w\|_{L^\infty} \leq \|g_1 - g_2\|_{L^\infty} + \varepsilon. \tag{193}$$

Sending $\varepsilon \to 0^+$, we have

$$\|u_1 - u_2\|_{L^\infty} \leq \|g_1 - g_2\|_{L^\infty}. \tag{194}$$

Hence, given any $\varepsilon > 0$, pick $\delta = \varepsilon$ such that $\|g_1 - g_2\|_{L^\infty} < \delta$. We then have by (194),

$$\|u_1 - u_2\|_{L^\infty} \leq \|g_1 - g_2\|_{L^\infty} < \delta = \varepsilon, \tag{195}$$

as required.

**Exercise 46.** (Spring 23 Problem 8 and Fall 21 Problem 4, Modified.) Let $u(x,t)$ solve the initial value problem:

$$\begin{cases} u_{tt} - 2u_{xt} - 15u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = \phi(x) & \text{on } \mathbb{R} \times \{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } \mathbb{R} \times \{t=0\}, \end{cases} \tag{196}$$

(i) Derive a formula for $u$ in terms of $\phi$ and $\psi$, when $\phi, \psi$ are $C^2$.

(ii) Next, consider the boundary value problem below:

$$\begin{cases} u_{tt} - 2u_{xt} - 15u_{xx} - u_t + 2024u = f(x,t) & \text{in } (0,1) \times (0,\infty), \\ u(0,t) = a(t) & \text{on } \{x=0\} \times [0,\infty), \\ u(1,t) = b(t) & \text{on } \{x=1\} \times [0,\infty), \\ u(x,0) = \phi(x) & \text{on } [0,1] \times \{t=0\}, \\ u_t(x,0) = \psi(x) & \text{on } [0,1] \times \{t=0\}, \end{cases} \tag{197}$$

with $a(t), b(t), \phi(x), \psi(x)$, and $f(x,t)$ sufficiently smooth.

Show that a smooth solution to (197) must be unique.

Hint: Consider the most general form of the associated energy:

$$E(t) = \int_0^1 (u_t)^2(x,t) + \alpha (u_x)^2(x,t) + \beta u^2(x,t) \mathrm{d}x$$

for some appropriate choice of positive constants $\alpha$ and $\beta$.

Suggested Solutions:

(i) Observe that we can factorize the operator into

$$\left(\frac{\partial}{\partial t} - 5\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x}\right) u = 0.$$

One can then show by a change of coordinates that the general solution is given by

$$u(x,t) = f(x+5t) + g(x-3t).$$

Plugging both initial conditions, we have

$$\begin{cases} f(x) + g(x) & = \phi(x), \\ 5f'(x) - 3g'(x) & = \psi(x). \end{cases}$$

Integrate the second equation from $0$ to obtain

$$5f(x) - 3g(x) = 5f(0) - 3g(0) + \int_0^x \psi(s)\mathrm{d}s$$

Solving these two equations simultaneously, we have

$$\begin{cases} f(x) & = \frac{5f(0)-3g(0)}{8} + \frac{1}{8}\int_0^x \psi(s)\mathrm{d}s + \frac{3}{8}\phi(x) \\ g(x) & = -\frac{5f(0)-3g(0)}{8} - \frac{1}{8}\int_0^x \psi(s)\mathrm{d}s + \frac{5}{8}\phi(x) \end{cases}$$

and hence

$$u(x,t) = \frac{3}{8}\phi(x+5t) + \frac{5}{8}\phi(x-3t) + \frac{1}{8}\int_{x-3t}^{x+5t} \psi(s)\mathrm{d}s.$$

(ii) Let $u_1$ and $u_2$ be solutions to (197). Then $w := u_1 - u_2$ satisfies

$$\begin{cases} w_{tt} - 2w_{xt} - 15w_{xx} - w_t + 2024w = 0 & \text{in } (0,1) \times (0,\infty), \\ u(0,t) = 0 & \text{on } \{x=0\} \times [0,\infty), \\ u(1,t) = 0 & \text{on } \{x=1\} \times [0,\infty), \\ u(x,0) = 0 & \text{on } [0,1] \times \{t=0\}, \\ u_t(x,0) = 0 & \text{on } [0,1] \times \{t=0\}, \end{cases}$$

Consider the energy

$$E(t) = \int_0^1 (w_t)^2 + \alpha(w_x)^2 + \beta w^2(x,t)\mathrm{d}x$$

for some choice of positive constants $\alpha$ and $\beta$ to be determined later. Compute

$$\begin{aligned} E'(t) &= \int_0^1 w_t w_{tt} + \alpha w_x w_{xt} + \beta w w_t \mathrm{d}x \\ &= \int_0^1 w_t(2w_{xt} + 15w_{xx} + w_t - 2024w) + \alpha w_x w_{xt} + \beta w w_t \mathrm{d}x \\ &= \int_0^1 (w_t)^2 \mathrm{d}x + \int_0^1 15 w_{xx} w_t + \alpha w_x w_{xt} \mathrm{d}x + \int_0^1 (\beta - 2024) w w_t \mathrm{d}x. \end{aligned}$$

Pick $\alpha = 15$ and $\beta = 2024$, Observe that the second integral above becomes $0$ upon integrating by parts and utilizing the fact that the boundary term vanishes. This implies that

$$E'(t) = \int_0^1 (w_t)^2 \mathrm{d}x \le E(t).$$

By Grönwall's inequality, we have

$$E(t) \le E(0)e^t \quad \text{for each } t > 0.$$

Since $E(0) = \int_0^1 (w_t)^2(x,0) + 15(w_x)^2(x,0) + 2024w^2(x,0) = 0$ by the initial data ($w(x,0) = w_t(x,0) = 0$, which upon differentiating with respect to $x$, also gives $w_x(x,0) = 0$). Hence, we have

$$0 \le E(t) = 0$$

for all $t > 0$ and thus $E(t) = 0$. This implies that $w \equiv 0$ (since $w$ appears directly in the energy and each term of the energy is non-negative as the constants picked are positive!) and hence $u_1 = u_2$ for all $t > 0$ and $x \in [0,1]$.

**Exercise 47.** (Spring 21, Problem 4.) A field $u : [0,\infty)^2 \times [0,\infty) \to \mathbb{R}$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2},$$

with boundary conditions : $u = 0$ on $x_2 = 0$ and $\frac{\partial u}{\partial x_1} = 0$ on $x_1 = 0$.

Suppose that at $t = 0$, the functions $u(\cdot, 0)$ and $\frac{\partial u}{\partial t}(\cdot, 0)$ are compactly supported on $\sqrt{x_1^2 + x_2^2} < 1$. Find the compact support for $u$ at time $t$.

If you make use of results for the domain of dependence of a solution of the wave equation, then you should prove them.

Hint: Do relevant extensions of the initial data and prove a domain of dependence result in 2D.

Suggested Solutions:

Since $u((x_1, 0), t) = 0$ for all $t > 0$ (and $x_1 > 0$), by differentiating with respect to time, we have $u_t((x_1, 0), t) = 0$ for all $t > 0$ too. Similarly, we have $\frac{\partial u}{\partial x_1}((0, x_2), t) = 0$ and $\frac{\partial^2 u}{\partial x_1 \partial t}((0, x_2), t) = 0$ too. Hence, we consider the odd extension of the initial data across $x_2 = 0$ and the even extension of the initial data across $x_1 = 0$ (here, initial data refers to both $u(\cdot, 0)$ and $\frac{\partial u}{\partial t}(\cdot, 0)$). Hence, it suffices to consider the initial value problem:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2},$$

with initial data $u(\cdot, 0)$ and $\frac{\partial u}{\partial t}(\cdot, 0)$ compactly supported on $\sqrt{x_1^2 + x_2^2} < 1$ with $(x_1, x_2) \in \mathbb{R}^2$. (The compact support of this new equation is the compact support of the origin equation on the first quadrant upon restricting the solution to the first quadrant.)

Claim: The compact support is given by $B(0, 1 + t)$ (an open ball of radius $1 + t$ centered at $0$ at a given time $t$). To prove this, it suffices to prove the following domain of dependence result:

**Lemma: (Domain of Dependence.)** Fix any $(x_0, t_0) \in \mathbb{R}^2 \times [0, \infty)$. If $w = w_t \equiv 0$ on $\{t = 0\} \times B(x_0, t_0)$, then we have that $w \equiv w_t \equiv 0$ in the backwards cone $K(x_0, t_0)$ given by

$$K(x_0, t_0) := \{(x, t) \in \mathbb{R}^2 \times [0, t_0] : |x - x_0| \leq (t_0 - t)\}.$$

Here, $w$ solves the wave equation:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2}.$$

In other words, the speed of propagation is given by $1$.

Consider the local energy defined by

$$e(t) = \int_{B(x_0, t_0 - t)} w_t^2(x, t) + |\nabla w|^2(x, t) \mathrm{d}x.$$

Recall that $|\nabla w|^2 = \nabla w \cdot \nabla w = \left(\frac{\partial w}{\partial x_1}\right)^2 + \left(\frac{\partial w}{\partial x_2}\right)^2$. By applying the Reynold's transport theorem, we obtain

$$e'(t) = \int_{B(x_0, t_0 - t)} \frac{\partial}{\partial t}\left(w_t^2(x, t) + |\nabla w|^2(x, t)\right) \mathrm{d}x + \int_{\partial B(x_0, t_0 - t)} w_t^2(x, t) + |\nabla w|^2(x, t)(\mathbf{n}(x) \cdot \mathbf{V}(x)) \mathrm{d}S(x),$$

where $\mathbf{V}(x)$ is the velocity vector on a point on the surface, and $\mathbf{n}(x)$ is the unit outward-pointing normal vector.

The first integral can be computed to obtain

$$\begin{aligned}
\int_{B(x_0,t_0-t)} \frac{\partial}{\partial t}\left(w_t^2+|\nabla w|^2\right)\mathrm{d}x &= \int_{B(x_0,t_0-t)} w_t w_{tt} + \nabla w\cdot\nabla(w_t)\mathrm{d}x \\
&\underbrace{=}_{\text{Wave Equation}} \int_{B(x_0,t_0-t)} w_t(\Delta w) + \nabla\cdot(w_t\nabla w) - w_t\Delta w\mathrm{d}x \\
&= \int_{B(x_0,t_0-t)} \nabla\cdot(w_t\nabla w)(x,t)\mathrm{d}x \\
&\underbrace{=}_{\text{Divergence Theorem}} \int_{\partial B(x_0,t_0-t)} w_t(x,t)\nabla w(x,t)\cdot\mathbf{n}(x)\mathrm{d}S(x) \\
&\le \int_{\partial B(x_0,t_0-t)} |w_t||\nabla w(x,t)|\mathrm{d}S(x).
\end{aligned}$$

The last inequality is obtained by using Cauchy-Schwarz and the fact that $\mathbf{n}$ has magnitude $1$ since it is the unit normal to the surface of the ball. On the other hand, the second term can be computed by

$$\begin{aligned}
&= \int_{\partial B(x_0,t_0-t)} \left(w_t^2(x,t)+|\nabla w|^2(x,t)\right)(\mathbf{n}(x)\cdot\mathbf{V}(x))\mathrm{d}S(x) \\
&= -\int_{\partial B(x_0,t_0-t)} \left(w_t^2(x,t)+|\nabla w|^2(x,t)\right)\mathrm{d}S(x).
\end{aligned}$$

Here, we have used the fact that the unit normal is pointing in the radial direction and in the opposite direction as the velocity vector, and the velocity vector has magnitude $1$ (as obtained by computing the speed at which the boundary moves inwards). Combining the two terms, we thus have

$$e'(t) \le \int_{\partial B(x_0,t_0-t)} |w_t(x,t)||\nabla w(x,t)| - \frac{w_t^2(x,t)}{2} - \frac{|\nabla w|^2(x,t)}{2}\mathrm{d}S(x) \le 0$$

by Cauchy-Schwarz inequality, since $|w_t||\nabla w| \le \frac{w_t^2}{2}+\frac{|\nabla w|^2}{2}$. Hence, since $e'(t)\le 0$ and hence $0\le e(t)\le e(0)=0$. This implies that $w_t\equiv w_x\equiv 0$ on the cone $K(x_0,t_0)$, and by employing the standard arguments, one can deduce that $w\equiv 0$ on the cone $K(x_0,t_0)$. This concludes the proof of the lemma and justifies the claim on the support of $u$ at time $t$. For the original problem, by restricting the solution obtained to the first quadrant, we thus have that the support is given by

$$B(0,1+t)\cap\{(x_1,x_2): x_1\ge 0, x_2\ge 0\}.$$

**Exercise 48.** (Spring 21, Problem 5 (i).) Consider the conservation law

$$\begin{cases} u_t + (u^3)_x = 0 & \text{in } \mathbb{R} \times (0, \infty); \\ u = g & \text{in } \mathbb{R} \times \{t = 0\}. \end{cases}$$

We refer to *entropy solutions* as integral solutions which satisfy the Lax entropy condition.

Find an entropy solution when

$$g(x) = 1 \quad \text{for } 0 < x < 1, \text{ otherwise } g = 0.$$

Suggested Solutions:

For small $t > 0$, we have a shock forming around $x = 1$, and a rarefaction fan forming around $x = 0$. The shock has speed stipulated by the Rankine-Hugoniot condition as:

$$s = \frac{u_+^3 - u_-^3}{u_+ - u_-} = u_+^2 + u_+ u_- + u_-^2 = 1^2$$

since the left state has value $1$ and the right state is at $0$. The rarefaction fan's formula can be determined using

$$f'(u) = 3u^2, \quad (f')^{-1}(u) = \sqrt{\frac{u}{3}} \text{ for } u > 0.$$

Hence, for small time (in fact, $t \in (0, 0.5)$), we have

$$u(x,t) = \begin{cases} 0 & \text{for } x < 0. \\ \sqrt{\frac{x}{3t}} & \text{for } 0 < x < 3t, \\ 1 & \text{for } 3t < x < t + 1, \\ 0 & \text{for } x > t + 1. \end{cases}$$

From the formula above, we observe that the right end of the rarefaction fan at $x = 0$ collides with the shock curve when $3t = t + 1$, ie $t = \frac{1}{2}$. Afterwards, the left state feeding into the shock curve originates from the rarefaction fan.

Parameterize the resulting shock curve for $t > 1/2$ by $(\gamma(t), t)$. The right state feeding into the shock curve remains unchanged at $u = 0$ (since it comes from $u = 0$ from $x > 1$), while the left state is given by

$$u_- = \sqrt{\frac{\gamma(t)}{3t}}$$

from the rarefaction fan formula above. By the Rankine-Hugoniot jump condition, we have that the shock speed satisfies (since $u_+ = 0$)

$$\gamma'(t) = (u_-)^2 = \frac{\gamma(t)}{3t}$$

with initial condition $\gamma(0.5) = 1.5$ (here, $1.5$ is obtained as the value of $x$ during the time of collision, ie $3 \times 0.5$ or $0.5 + 1$). The solution to the ODE above is given by

$$\gamma(t) = \frac{3}{2^{2/3}} t^{1/3}.$$

Henceforth, the solution for $t > 1/2$ is given by

$$u(x,t) = \begin{cases} 0 & \text{for } x < 0. \\ \sqrt{\frac{x}{3t}} & \text{for } 0 < x < \gamma(t), \\ 0 & \text{for } x > \gamma(t), \end{cases}$$

with $\gamma(t)$ defined above.

# References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