MATH32BH - Discussion Supplements for Winter 22.

Some of the contents are motivated from [1], [2], and [3] (in decreasing order of reference) and the lecture notes for Math 32BH (for Winter 22) by Richard Wong. [1]

Contents

1 Discussion 1

Summary for Lectures 1 and 2

Definition 1. Let $A \subset \mathbb{R}^n$. The **indicator** function 1_A is the function defined by

$$1_A(\mathbf{x}) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \tag{1}$$

Definition 2. The norm $||.||$ on $\mathbb{R}^n$ is defined to be the standard Euclidean norm on $\mathbb{R}^n$. Mathematically, for $\mathbf{x} = (x_1, ..., x_n)$ for a positive integer n, we have

$$||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}. \tag{2}$$

Definition 3. An open ball in $\mathbb{R}^n$ centered at $\mathbf{x}$ with radius $r \geq 0$ is denoted by $B_r(\mathbf{x})$. Mathematically,

$$B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{y}|| < r\}. \tag{3}$$

Definition 4. Let $A \subset \mathbb{R}^n$. We say that the subset A is **bounded** if there exists $r > 0$ such that $A \subset B_r(\mathbf{0})$.

Definition 5. A function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is **bounded** if its image

$$\{f(\mathbf{x}) : \mathbf{x} \in A\} \tag{4}$$

is a bounded subset of $\mathbb{R}$.

Definition 6. A point $\mathbf{x} \in \mathbb{R}^n$ is a **boundary point** of $A \subset \mathbb{R}^n$ if for any $\varepsilon > 0$, we have

1. $B_\varepsilon(\mathbf{x}) \cap A$ is non-empty and
2. $B_\varepsilon(\mathbf{x}) \cap A^c$ is non-empty

Notation: We denote

$$\partial A := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \text{ is a boundary point of } A\} \tag{5}$$

as the boundary of A.

Definition 7. The **closure** of the set $A \subset \mathbb{R}^n$, denoted by $\overline{A}$, is the union of A and the boundary of A.
Mathematically,

$$\overline{A} = A \cup \partial A. \tag{6}$$

Alternatively, one can characterize the closure of the set A as

$$\overline{A} = \{\mathbf{x} \in \mathbb{R}^n : \forall r > 0, B_r(\mathbf{x}) \cap A \neq \emptyset\}. \tag{7}$$

Definition 8. Let $A \subset \mathbb{R}^n$. We say that the set A is **closed** if

$$A = \overline{A}. \tag{8}$$

Note the alternative characterization of closed sets in 32AH:

Definition 9. We say that a point $\mathbf{x} \in X$ is a **limit point** if there is a sequence $\{\mathbf{x}_i\}_i$ contained inside X such that $\mathbf{x}_i$ converges to $\mathbf{x}$.

Definition 10. Let $A \subset \mathbb{R}^n$. We say that the set A is **closed** if and only if it contains all of its limit points. Mathematically, A possess the following property

$$\text{If } \mathbf{x}_i \xrightarrow{i\to\infty} \mathbf{x} \text{ and } \{\mathbf{x}_i\}_i \subset A, \text{ then } \mathbf{x} \in A. \tag{9}$$

(This is because the limiting point $\mathbf{x}$ might only be in $\mathbb{R}^n$ and not necessarily be in A. Example: $A = (0, \infty) \subset \mathbb{R}$, take the sequence $x_n = \frac{1}{n}$ and note that $0 \notin A$.)

Definition 11. The **support** of a function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is the <u>closure</u> of the set

$$\{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \neq 0\}. \tag{10}$$

Mathematically, if we denote the support of a function as $\text{supp}(f)$, it is thus defined as

$$\text{supp}(f) = \overline{\{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \neq 0\}}. \tag{11}$$

Definition 12. In addition, we say that a function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ has **bounded support** if the support of the function f is bounded. Equivalently, there exists $R > 0$ such that $f(\mathbf{x}) = 0$ for $||\mathbf{x}|| > R$.

Example 13. Let $A = \{x \in \mathbb{R} : x \text{ is even}\}$ and consider $f(x) = 1_A(x)$ for $f : \mathbb{R} \to \mathbb{R}$

- The set $A = \{..., -4, -2, 0, 2, 4, ...\}$ is not bounded.
- The image of f, given by $\{f(x) : x \in A\} = \{0, 1\}$, is a bounded set. Thus, the function f is bounded.
- One can check that $\partial A = \{..., -4, -2, 0, 2, 4, ...\} = A$.
  <u>$x \notin A$</u>: For x not in A, we can find an ε such that $B_\varepsilon(x) \cap A = \emptyset$.
  (Example: If $x = 1.99$, we pick $\varepsilon = 0.005$ and see the $B_{0.005}(1.99) = (1.985, 1.995)$ and thus does not intersect A.)
  <u>$x \in A$</u>: One can check that for any $\varepsilon > 0$, we have $B_\varepsilon(x) \cap A \supset \{x\} \neq \emptyset$. Furthermore, $B_\varepsilon(x) \cap A^c$ is non-empty as the region outside of x is basically A^c and thus the open ball centered about x will intersect with some points in A^c.
- This implies that the closure of A is given by
  $$\overline{A} = A \cup \partial A = A \cup A = A.$$
- By the above point, A is a closed set.
- Since A, the support of f, is not bounded, we then we say that f does not have a bounded support.

As mentioned in the lectures, for integrals of some function f over a set, say A, we can define $g(\mathbf{x}) = 1_A(\mathbf{x})f(\mathbf{x})$ and this is thus a function on $\mathbb{R}^n$. Mathematically, we have

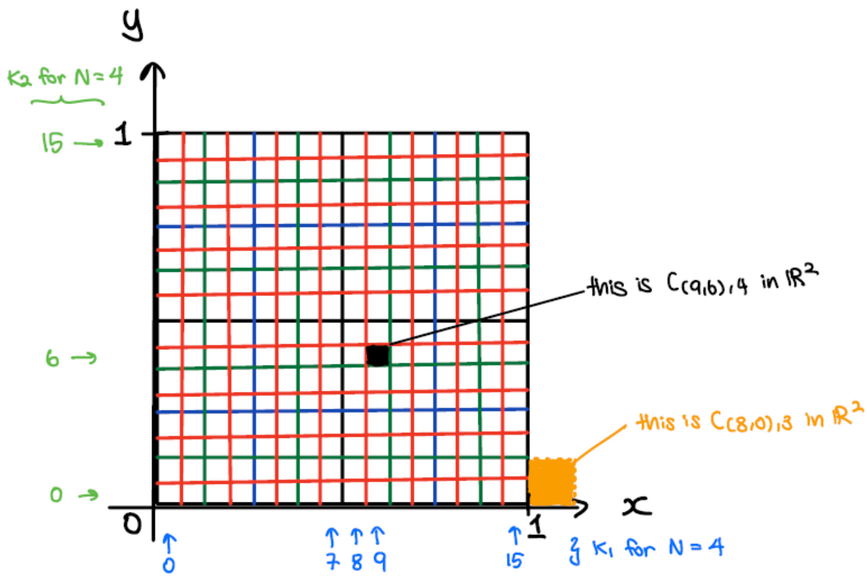
$$\int_A f(\mathbf{x})\mathrm{d}V = \int_{\mathbb{R}^n} 1_A(\mathbf{x})f(\mathbf{x})\mathrm{d}V = \int_{\mathbb{R}^n} g(\mathbf{x})\mathrm{d}V.$$

It is thus sufficient for us to be able to define integrals on $\mathbb{R}^n$. This is done by partitioning $\mathbb{R}^n$ into dyadic (hyper) cubes, as explained below.

Definition 14. Given a vector $\mathbf{k} = (k_1, ..., k_n) \subset \mathbb{Z}^n$, the **dyadic cube** $C_{\mathbf{k},N}$ is denoted by

$$C_{\mathbf{k},N} = \{\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n : \frac{k_i}{2^N} \leq x_i < \frac{k_1 + 1}{2^N} \text{ for } 1 \leq i \leq n\}. \tag{12}$$

Intuitively, N represents the number of dyadic ("halfing") divisions on a cube of side length 1. The following diagram in $\mathbb{R}^2$ illustrates this.



- The first dyadic level ($N = 1$) corresponds to the black lines, cutting the square into 4 equal squares of side length $\frac{1}{2}$.
- The next 3 dyadic levels for $N = 2, 3,$ and 4 are given by blue, green and red lines respectively.
- The black cube corresponds to the dyadic cube $C_{(9,6),4}$ in $\mathbb{R}^2$, since it is the 10^{th} cube counting from the left and the 7^{th} cube counting from the bottom for $N = 4$. Matheamtically, we have

$$C_{(9,6),4} = \left\{(x, y) \in \mathbb{R}^2 : \frac{9}{2^4} = \frac{9}{16} \leq x < \frac{10}{16}, \frac{6}{16} \leq y < \frac{7}{16}\right\}.$$

- Note that even though we are looking at a cube of side length 1, we are able to cover the whole of $\mathbb{R}^n$ based on our definition of dyadic cubes. For instance, the orange cube in the diagram above corresponds to $C_{(8,0),3}$. We can pick components of $\mathbf{k}$ to be greater than $2^N - 1$ (in this case, $2^3 - 1$) since we allow the components of $\mathbf{k}$ to be any arbitrary integer ($\mathbf{k} \in \mathbb{Z}^n$).

Proposition 15. The volume of a dyadic cube $C_{\mathbf{k},N}$ in $\mathbb{R}^n$, denoted by $\text{vol}(C_{\mathbf{k},N})$, is $\frac{1}{2^{Nn}}$.

Proposition 16. For a fixed N, the collection of all dyadic cubes $D_N(\mathbb{R}^n)$, also known as the N**-th dyadic partition of** $\mathbb{R}^n$, is given by

$$D_N(\mathbb{R}^n) := \{C_{\mathbf{k},N} : \mathbf{k} \in \mathbf{Z}^n\}. \tag{13}$$

Note that as the name suggests, $D_N(\mathbb{R}^n)$ partitions $\mathbb{R}^n$.

Definition 17. Let $X \subset \mathbb{R}$. (Replace all underlined words with words in blue for the corresponding definitions for infimums and lower bounds.)

- A number $a \in \mathbb{R}$ is an **upper bound** **lower bound** for X if for every $x \in X$, we have $x \leq a$ $(x \geq a)$
- A number $b \in \mathbb{R}$ is the **supremum** **infimum** for X if for every upper bound (lower bound) a of X, we have $b \leq a$ $(b \geq a)$.
  Note that the supremeum (infimum) can be understood as the least upper bound (greatest lower bound).
- We write $b := \sup(X)$ $(b := \inf(X))$. If X is not bounded from above (bounded from below), then we write $\sup(X) = \infty$ $(\inf(X) = -\infty)$.

Theorem 18. (Completeness of $\mathbb{R}$.) Every non-empty subset $X \subset \mathbb{R}$ has a supremum and infimum. Moreover, $\sup(X)$ and $\inf(X)$ are unique.

Remark: For students who have/are taking a class in Analysis (131 series), the standard "completeness of $\mathbb{R}$" talks about a non-empty subset **bounded from above** **(bounded from below)** has a supremum (infimum) in $\mathbb{R}$. However, since we are allowing sup and inf to take the value of ∞ and $-\infty$, the third bullet point in Definition 17 takes care of the case in which the subset X is unbounded.

Example 19. Essentially, inf and sup acts as the minimum and maximum to a large extent. However, the minimum and maximum of a set requires that the element is in the set itself. Consider the set $X = [1, 2)$ and $Y = (0, \infty)$.

- $\inf(X) = 1$ since 1 is a lower bound for X and for any other lower bounds for X, 1 will be greater or equals to that. Similarly, we have $\sup(X) = 2$, $\inf(Y) = 0$, and $\sup(Y) = \infty$.
- Note that if we were to ask for $\max(X)$, this is equivalent to some $a \in X$ such that $a \geq x$ for all $x \in X$. Intuitively, you would want to pick 2. However, 2 is not in the set. If you pick any other number that is smaller than 2, say 1.999, this is definitely an upper bound for X since 1.9999 is in X and is larger than 1.999.

With that, we can define sums that approximate the integral of a bounded function with bounded support f as follows.

Definition 20. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $A \subset \mathbb{R}^n$ be an arbitrary subset. Then, we define the following quantities:

$$M_A(f) := \sup(\{f(\mathbf{x}) : \mathbf{x} \in A\}), \quad m_A(f) := \inf(\{f(\mathbf{x}) : \mathbf{x} \in A\}). \tag{14}$$

Definition 21. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. The N**-th upper** and **lower** sums of f are defined as

$$U_N(f) := \sum_{C \in D_N(\mathbb{R}^n)} M_C(f) \operatorname{vol}(C), \quad L_N(f) := \sum_{C \in D_N(\mathbb{R}^n)} m_C(f) \operatorname{vol}(C) \tag{15}$$

By Proposition 15, since $\text{vol}(C) = \frac{1}{2^{nN}}$, we have

Proposition 22.

$$U_N(f) := \frac{1}{2^{nN}} \sum_{C \in D_N(\mathbb{R}^n)} M_C(f), \quad L_N(f) := \frac{1}{2^{nN}} \sum_{C \in D_N(\mathbb{R}^n)} m_C(f). \tag{16}$$

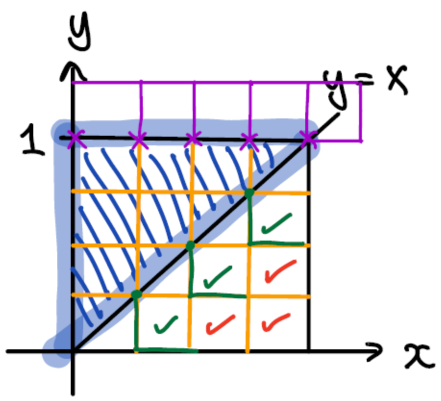
Furthermore, we have

Proposition 23. $U_N(f)$ is non-increasing (decreasing or constant) with respect to N (as N increases) and $L_N(f)$ is non-decreasing (increasing or constant) with respect to N (as N increases).

Intutively, this makes sense as the "overestimation" of the integral by $U_N(f)$ either decreases or stays the same as N increases. Similarly, the "underestimation" of the integral by $L_N(f)$ either decreases or stays (thus $L_N(f)$ either increases or stays constant) as N increases.
Let us look at an example as to how we can compute some of the N-th upper and lower sums.

Example 24. Denote $A = \{(x, y) \in \mathbb{R}^2 : y \geq x \text{ and } 0 \leq x, y \leq 1\}$ and the function $f(x, y) = 1_A(x, y)$. Compute $U_2(f)$ and $L_2(f)$. Use these to estimate $\int_A f(x, y)\mathrm{d}A$.

Suggested Solution: Since $A \subset [0, 1] \times [0, 1]$, we can consider a square with side length 1. Note that since $f = 1_A$, the image of f is the set $\{0, 1\}$ and is thus a bounded set. Furthermore, the support of f is a subset of A, which is a subset of $[0, 1] \times [0, 1]$, a bounded subset of $\mathbb{R}^2$. Thus, f is a bounded function of bounded support. This implies that the upper sum and lower sum makes sense by Definition 21. From the same definition, we can compute $U_2(f)$ and $L_2(f)$ with the help of the following diagram as follows.



At the dyadic level $N = 2$, we have a total of $2^{nN} = 2^4 = 16$ squares. Note that $f(x) = 1$ for x in the blue shaded region corresponding to $y \geq x$, and including the diagonal line $y = x$ and the two boundary lines $y = 1$ and $x = 0$.

Note that by definition of a dyadic cube (square in this case) as in Definition 14, we see that for each square, the left and the bottom boundaries are included with only the bottom left corner of the square included. In view of that, we will be including the 5 additional purple squares above the line $y = 1$.
With that, we can characterize the 21 squares into the following:

- There are 3 squares labelled with a red tick in the graph above. Note that in these squares B_i, $f(x, y) = 0$ on the entire squares. Thus,

$$m_{B_i}(f) = M_{B_i}(f) = 0.$$

- There are a total of 6 squares C_i in the **shaded region** in which the line $y = x$ does not cut across into them directly. In these squares, $f(x) = 1$. Thus,

$$m_{C_i}(f) = M_{C_i}(f) = 1.$$

- There are a total of 3 squares D_i labelled with a green tick in the graph above. Thus, the included boundary misses $y = x$. Henceforth,

$$m_{D_i}(f) = 0, M_{D_i}(f) = 0.$$

- Similarly, there are 4 squares E_i along the diagonal in which $y = x$ bisects each of them diagonally. For each of these sqaures, there are points inside and outside of the shaded region. This implies that

$$m_{E_i}(f) = 0, M_{E_i}(f) = 1.$$

- For the 5 purple squares F_i, note that the bottom left corner of the square lies on the line $y = 1$, in which $f(x, y) = 1$ along this line. This implies that

$$m_{F_i}(f) = 0, M_{F_i}(f) = 1.$$

By proposition 22, we have

$$U_2(f) = \frac{15}{16}, \quad \text{and} \quad L_2(f) = \frac{6}{16}. \tag{17}$$

since there are 15 squares with $M_{\text{each square}}(f) = 1$ and 6 squares with $m_{\text{each square}}(f) = 1$. An estimate for $\int_A f(x, y)\mathrm{d}A$ would be say the average of these two quantities, at $\approx \frac{21}{32}$. Since this is not exactly rigorous, any value in between $L_2(f)$ and $U_2(f)$ would work given sufficient justification.

Remark: The problem would be greatly simplified if the domain excludes the line $y = 1$ (and possibly $x = 0$).

Exercises:

Exercise 1. Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \text{ and } x \geq 0\}$. Compute the following:

(i) ∂A, and

(ii) $\overline{A}$.

Exercise 2. Let us consider dyadic cubes $C_{\mathbf{k},N}$ in $\mathbb{R}^3$. Determine the value of $\mathbf{k}$ such that $\left(\frac{31}{32}, \frac{65}{64}, 1.01\right) \in C_{\mathbf{k},5}$.

Exercise 3. Let A and B be subsets of $\mathbb{R}$ such that $A \subset B$. Prove that

(i) $\sup(A) \leq \sup(B)$,

(ii) $\sup(-A) = -\inf(A)$, and

(iii) $\inf(A) \geq \inf(B)$.

Exercise 4. Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and let $f(x) = 1_A(x)$.
Compute the values of $U_1(f)$ and $L_1(f)$.

Partial Solutions/Hints:

- Exercise 1. (i) $\partial A = \{(x, y) \in \mathbb{R}^2 : (x^2 + y^2 = 1 \text{ and } x \geq 0) \text{ OR } (y = 0 \text{ and } -1 \leq x \leq 1)\}$, (ii) $\overline{A} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } x \geq 0\}$.
- Exercise 2. $\mathbf{k} = (31, 32, 32)$.
- Exercise 3. Expanding out the definitions for $\sup$ and $\inf$ should give you the required tools to work with. Note that (iii) follows from (i) and (ii) if you think hard enough about it.
- Exercise 4. $U_1(f) = 1$, $L_1(f) = \frac{4}{16}$.

2 Discussion 2

Summary for Lectures 3 - 5

Definition 25. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $A \subset \mathbb{R}^n$ be an arbitrary subset. Then, we define the following quantities:

$$M_A(f) := \sup(\{f(\mathbf{x}) : \mathbf{x} \in A\}), \quad m_A(f) := \inf(\{f(\mathbf{x}) : \mathbf{x} \in A\}). \tag{18}$$

Definition 26. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. The N**-th upper** and **lower** sums of f are defined as

$$U_N(f) := \sum_{C \in D_N(\mathbb{R}^n)} M_C(f)\,\mathrm{vol}(C), \quad L_N(f) := \sum_{C \in D_N(\mathbb{R}^n)} m_C(f)\,\mathrm{vol}(C) \tag{19}$$

Since $\mathrm{vol}(C) = \frac{1}{2^{nN}}$ (see Discussion Supplement 1), we have

Proposition 27. Let f to be given as in Definition 21. Then,

$$U_N(f) := \frac{1}{2^{nN}} \sum_{C \in D_N(\mathbb{R}^n)} M_C(f), \quad L_N(f) := \frac{1}{2^{nN}} \sum_{C \in D_N(\mathbb{R}^n)} m_C(f). \tag{20}$$

Next, we define the upper integral $U(f)$ and the lower integral $L(f)$ as follows.

Definition 28. Let f to be given as in Definition 21. The **upper** and **lower** integrals of f are defined as

$$U(f) := \lim_{N\to\infty} U_N(f), \quad L(f) := \lim_{N\to\infty} L_N(f). \tag{21}$$

Furthermore, we say that f is **integrable** if $U(f) = L(f)$, with the **integral** of f defined as

$$\int_{\mathbb{R}^n} f(\mathbf{x})\mathrm{d}V = U(f) = L(f). \tag{22}$$

Note that using Definition 21 to compute $U_N(f)$ and $L_N(f)$, and consequently $U(f)$ and $L(f)$, might be tedious and challenging. Thus, instead of finding $M_C(f)$ and $m_C(f)$, it suffices to pick a point $\mathbf{x}$ in each dyadic cube C. This motivates the following definitions and propositions relating to what is known as Riemann integrals.

Definition 29. The N-th Riemann sum is defined as

$$R_N(f) := \sum_{C \in D_N(\mathbb{R}^n)} f(\mathbf{x}_{\mathbf{k},N})\,\mathrm{vol}(C), \tag{23}$$

where $\mathbf{x}_{\mathbf{k},N}$ is any arbitrary point in a given dyadic cube $C \in D_N(\mathbb{R}^n)$.

Proposition 30. If $f : \mathbb{R}^n \to \mathbb{R}$ **is integrable**, then

$$\lim_{N\to\infty} R_N(f) = \int_{\mathbb{R}^n} f(\mathbf{x})\mathrm{d}V. \tag{24}$$

Note the following remarks:

- In general, Proposition 30 works only if we know that f is integrable.

- Proposition 30 allows us to choose whichever point we want in a given dyadic cube. This removes ambiguity from considering points along the boundary of the dyadic cubes (see Discussion Supplement 1 for an example of the aforementioned remark).
- The requirement that f is integrable is necessary. A common counterexample would be the **dirichlet** function on $[0, 1]$.
  Mathematically, it is given by the indicator function on the set $\mathbb{Q} \cap [0, 1]$ (Here, $\mathbb{Q}$ refers to the set of rational numbers). Explicitly,

$$1_{\mathbb{Q}\cap[0,1]}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

  One can see that, $1_{\mathbb{Q}\cap[0,1]}$ is a bounded function (since the output is either 0 or 1) with bounded support (since we are looking at a subset of $[0, 1$, which itself is bounded).

  However, one can show that $U_N(1_{\mathbb{Q}\cap[0,1]}) = 1$ and $L_N(1_{\mathbb{Q}\cap[0,1]}) = 0$ for any $N \in \mathbb{N}$. Thus,

$$U(1_{\mathbb{Q}\cap[0,1]}) = \lim_{N\to\infty} U_N(1_{\mathbb{Q}\cap[0,1]}) = \lim_{N\to\infty} 1 = 1, \tag{26}$$

  and

$$L(1_{\mathbb{Q}\cap[0,1]}) = \lim_{N\to\infty} L_N(1_{\mathbb{Q}\cap[0,1]}) = \lim_{N\to\infty} 0 = 0. \tag{27}$$

  This implies that $U(1_{\mathbb{Q}\cap[0,1]}) = 1 \neq 0 = L(1_{\mathbb{Q}\cap[0,1]})$ and thus is not an integrable function.

  From this, we can see that the Riemann sums depends on our choice our choice of $\mathbf{x}$, in the sense that it is possible to achieve $R(f) = \frac{1}{2}$ or in fact any real number between 0 and 1 inclusive.
  ($\frac{1}{2}$ can be attained such that for all the given dyadic cubes at a given dyadic level N, we can pick x to be a rational number for half of the cubes and x to be irrational for the other half of the cubes. Note that this is possible as any dyadic cube $C \subseteq \mathbb{R}$ contains infinitely many rational and irrational numbers.)
  This implies that Proposition 30 fails.

Theorem 31. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two integrable functions. Then, we have the following properties.

(i) $f + g$ is integrable, and

$$\int_{\mathbb{R}^n} (f+g)(\mathbf{x})\mathrm{d}V = \int_{\mathbb{R}^n} f(\mathbf{x})\mathrm{d}V + \int_{\mathbb{R}^n} g(\mathbf{x})\mathrm{d}V. \tag{28}$$

(ii) If $\lambda \in \mathbb{R}$, then λf is integrable and

$$\int_{\mathbb{R}^n} \lambda f(\mathbf{x})\mathrm{d}V = \lambda \int_{\mathbb{R}^n} f(\mathbf{x})\mathrm{d}V. \tag{29}$$

(iii) If $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} f(\mathbf{x})\mathrm{d}V \leq \int_{\mathbb{R}^n} g(\mathbf{x})\mathrm{d}V. \tag{30}$$

(iv) (Triangle Inequality) $|f|(\mathbf{x}) := |f(\mathbf{x})|$ is integrable with

$$\left|\int_{\mathbb{R}^n} f(\mathbf{x})\mathrm{d}V\right| \leq \int_{\mathbb{R}^n} |f|(\mathbf{x})\mathrm{d}V \tag{31}$$

(v) Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ be two integrable functions. Then, $h : \mathbb{R}^{n+m} \to \mathbb{R}$ with $h(\mathbf{x}, \mathbf{y}) = f(\mathbf{x})g(\mathbf{y})$ defines an integrable function on $\mathbb{R}^{n+m}$, with

$$\int_{\mathbb{R}^{n+m}} h(\mathbf{x},\mathbf{y})\mathrm{d}V = \int_{\mathbb{R}^n} f(\mathbf{x})\mathrm{d}V \int_{\mathbb{R}^m} g(\mathbf{y})\mathrm{d}V \tag{32}$$

Definition 32. For any function $f : \mathbb{R}^n \to \mathbb{R}$, we define two auxillary functions f^+ and f^- as follows:

$$f^+(\mathbf{x}) = f(\mathbf{x})1_{\{\mathbf{x}\in\mathbb{R}^n : f(\mathbf{x})\geq 0\}}(\mathbf{x}), \quad f^-(\mathbf{x}) = -f(\mathbf{x})1_{\{\mathbf{x}\in\mathbb{R}^n : f(\mathbf{x})\leq 0\}}(\mathbf{x}). \tag{33}$$

Furthermore, note that

$$f = f^+ - f^-. \tag{34}$$

Intuitively, Definition 32 decomposes f into the positive and the negative portions. For the negative "portion", the sign of it is flipped so that both $f^+(\mathbf{x}) \geq 0$ and $f^-(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

Corollary 33. A bounded function f with bounded support is integrable if and only if both f^+ and f^- are integrable.

For practice purposes, we shall try to prove the corollary above, below.

Proof:
$(\to)$. Suppose that f is integrable.

To show that f^+ is integrable,

- By (iv) of Theorem 31, we have that $|f|$ is integrable.
- Note that we can write f^+ in terms of f as follows:

$$f^+ = \frac{1}{2}(|f| + f), \quad f^- = \frac{1}{2}(|f| - f). \tag{35}$$

- Since $|f|$ is integrable and f is integrable (by assumption), then $|f| + f$ is integrable by (i) of Theorem 31.
- Since $\frac{1}{2} \in \mathbb{R}$, then by (ii) of Theorem 31, $\frac{1}{2}(|f| + f)$ is integrable.
- By the decomposition in (35), we have that f^+ is integrable.

It might be tempting to say that f^- is integrable by a "similar" argument. You will be right if you say that except for just one small detail; the negative sign! This is *italicized* in the proof below.
To show that f^- is integrable,

- By (iv) of Theorem 31, we have that $|f|$ is integrable.
- From (35), we have the corresponding decomposition of f^-.
- *Since f is integrable and $-1 \in \mathbb{R}$, we have that $(-1)f = -f$ is integrable.*
- Since $|f|$ is integrable and $-f$ is integrable (by assumption), then $|f| + (-f) = |f| - f$ is integrable by (i) of Theorem 31.
- Since $\frac{1}{2} \in \mathbb{R}$, then by (ii) of Theorem 31, $\frac{1}{2}(|f| - f)$ is integrable.
- By the decomposition in (35), we conclude f^- is integrable.

$(\leftarrow)$. For the other direction, we assume that f^+ and f^- are integrable.

- Since f^- is integrable, by (ii) of Theorem 31, since $-1 \in \mathbb{R}$, then $-f^-$ is integrable.
- Since f^+ and $-f^-$ are both integrable, we have $f = f^+ + (-f^-) = f^+ - f^-$ is integrable, as required.

Remark: Note that we are a little "chatty" with the proof here, but this illustrates what a detailed proof is (well, I could be more rigorous with saying that $a + (-b) = a - b$ by how this is defined for real numbers, but this should be sufficient). If we want our proof to be succinct, it is okay to leave out certain minor details, such as $-1 \in \mathbb{R}$ if you think that it is not important (this might not

be the case in an abstract algebra class, since it might not be clear that certain elements belong to the "set" that we are working with). However, "important" is always subjective and varies across fields in Mathematics, and the art of "choosing what to include" is a skill that you get better at as you take more math classes!
More definitions below:

Definition 34. Let $A \subset \mathbb{R}^n$. If $1_A : \mathbb{R}^n \to \mathbb{R}$ is integrable, then the $n-$**dimensional volume** of A, $\mathrm{vol}_n(A)$, is

$$\mathrm{vol}_n(A) := \int_{\mathbb{R}^n} 1_A \mathrm{d}V. \tag{36}$$

Definition 35. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, and let $A \subset \mathbb{R}^n$. The **oscillation** of f over A, $\mathrm{osc}_A(f)$. is defined as

$$\mathrm{osc}_A(f) := M_A(f) - m_A(f) = \sup_{\mathbf{x}\in A} f(\mathbf{x}) - \inf_{\mathbf{x}\in A} f(\mathbf{x}) \tag{37}$$

Theorem 36. A function $f : \mathbb{R}^n \to \mathbb{R}$ is integrable if and only if

(i) f is bounded with bounded support, and

(ii) For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{\{C\in D_N(\mathbb{R}^n):\mathrm{osc}_A(f)>\varepsilon\}} \mathrm{vol}_n\, C < \varepsilon. \tag{38}$$

Intuitively, (ii) reads as the "total volume" of dyadic cubes in which the oscillation is large in each of these dyadic cubes.
To prove the theorem that a continuous function with bounded support is integrable, we would need to introduce some of the key terminologies in Mathematical Analysis as follows.

Theorem 37. (Elegance is not required.) Let $\varepsilon > 0$, and let u be a function of ε such that $u(\varepsilon) \to 0$ as $\varepsilon \to 0$. We say that a sequence $\{\mathbf{x_i}\}_i \subset \mathbb{R}^n$ converges to some $\mathbf{x} \in \mathbb{R}^n$ if

- For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $||\mathbf{x}_n - \mathbf{x}|| < \varepsilon$.
- For every $\varepsilon > 0$. there exists $M \in \mathbb{N}$ such that for all $m \geq M$, we have $||\mathbf{x}_n - \mathbf{x}|| < u(\varepsilon)$.

Intuitively, for a given $\varepsilon > 0$, if you can show that $||\mathbf{x}_n - \mathbf{x}|| < 32\varepsilon$ or say $||\mathbf{x}_n - \mathbf{x}|| < \varepsilon^{\frac{1}{2021}}$ for instance, then this is sufficient to show that $\mathbf{x}_n$ converges to $\mathbf{x}$.
Next, we recall some definitions and properties of "continuity" below. (These are concepts covered in 32AH, but we shall recap them here for students who did not take 32AH.) For the definitions below, let n, m be positive integers that are possibly different.

Definition 38. Let $X \subset \mathbb{R}^n$. We say that a function $f : X \to \mathbb{R}^m$ is **continuous** at a point $\mathbf{x} \in X$ if it satisfies the following property:
For all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $\mathbf{y} \in X$,

$$0 < ||\mathbf{x} - \mathbf{y}|| < \delta \implies ||f(\mathbf{x}) - f(\mathbf{y})|| < \varepsilon. \tag{39}$$

Definition 39. Let $X \subset \mathbb{R}^n$. We say that a function $f : X \to \mathbb{R}^m$ is **continuous** (on the set X) if it satisfies the following property:
For all $\varepsilon > 0$ and $\mathbf{x} \in X$, there exists a $\delta > 0$ such that for all $\mathbf{y} \in X$,

$$0 < ||\mathbf{x} - \mathbf{y}|| < \delta \implies ||f(\mathbf{x}) - f(\mathbf{y})|| < \varepsilon. \tag{40}$$

Alternatively, we say that f is continuous on X if for every point $\mathbf{x} \in X$, f is continuous at $\mathbf{x}$

(Note the difference in the two definitions above; the former corresponds to continuity of f at a point, the latter corresponds to the continuity of f on a set!)

Definition 40. Let $X \subset \mathbb{R}^n$. We say that a function $f : X \to \mathbb{R}^m$ is **uniformly continuous** if it satisfies the following property:
For all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $\mathbf{x}, \mathbf{y} \in X$,

$$0 < ||\mathbf{x} - \mathbf{y}|| < \delta \implies ||f(\mathbf{x}) - f(\mathbf{y})|| < \varepsilon. \tag{41}$$

Here, the difference is that given an arbitrary $\varepsilon > 0$, our choice of δ can depend on $\mathbf{x}$ if all we have to show is that f is continuous. However, if we want to show that f is uniformly continuous, our choice of δ must not depend on $\mathbf{x}$.
(Thus the function is "uniformly" continuous since we do not have to "adjust" the value of δ if say we have some $\mathbf{x} \in X$ that happens to be troublesome.)

Theorem 41. (Continuity is equivalent to sequential continuity.) Let $X \subset \mathbb{R}^n$, $f : X \to \mathbb{R}^m$ be a function and $\mathbf{x} \in X$ be some arbitrary chosen point in X. Then, the following statements are equivalent:

- f is continuous at $\mathbf{x}$.
- For every sequence $\{\mathbf{x}_i\}$ converging to $\mathbf{x}$, the sequence $\{f(\mathbf{x}_i)\}$ converges to $f(\mathbf{x})$.

Analogously, one can combine Definition 40 with Theorem 41 to obtain the following.

Theorem 42. (Uniform continuity is equivalent to sequential uniform continuity.) Let $X \subset \mathbb{R}^n$, $f : X \to \mathbb{R}^m$ be a function. Then, the following statements are equivalent:

- f is uniformly continuous on X.
- For any two sequences $\{\mathbf{x}_i\}_i, \{\mathbf{y}_i\}_i \subset X$ such that $||\mathbf{x}_i - \mathbf{y}_i||$ converges to 0, then $||f(\mathbf{x}_i) - f(\mathbf{y}_i)||$ converges to 0 as i tends to ∞.

Next, we shall introduce the notion of compact subsets of $\mathbb{R}^n$. For the purpose of this class, the abstract definition is not required. Instead, we can identify compact subsets of $\mathbb{R}^n$ as subsets satisfying certain properties.

Theorem 43. (Bolzano-Weierstrass Theorem on $\mathbb{R}^n$.) Let $K \subset \mathbb{R}^n$. If K is **compact**, then any sequence in K has a convergent subsequence in K.
Mathematically, given any $\{\mathbf{x}_i\}_i \subset K$, there exists an $\mathbf{x} \in K$ and a subsequence $\{\mathbf{x}_{i_j}\}_j \subset K$ such that

$$\lim_{j\to\infty} \mathbf{x}_{i_j} = \mathbf{x}. \tag{42}$$

Theorem 44. (Heine-Borel Property on $\mathbb{R}^n$.) Let $K \subset \mathbb{R}^n$. We say that K is **compact** if and only if K is both closed and bounded.

(Recall the definition of **closed** in Discussion Supplement 1.)

Theorem 45. Let $K \subset \mathbb{R}^n$, K compact, and let $f : K \to \mathbb{R}^m$ be continuous. Then f is uniformly continuous on f.

This means if we know that f is continuous, and the domain is a compact subset of $\mathbb{R}^n$, we can upgrade the "continuity" property of f such that f is now uniformly continuous on K.
With that, we end off the following theorem:

Theorem 46. Let $f : \mathbb{R}^n \to \mathbb{R}$. If f is continuous with bounded support, then f is integrable.

With some of these definitions and theorems in, let us look at some examples.

Example 47. Let $f : (0, \infty) \to (0, \infty)$ be such that $f(x) = \frac{1}{x}$. Then, f is continuous on $(0, \infty)$ but not uniformly continuous on $(0, \infty)$.

Intuitively, although 0 is excluded from the domain, if we force ourselves to pick some x close enough to 0, it is hard to obtain the same bound "$|f(x) - f(y)| < \varepsilon$" using a δ that does not depend on x (since if x is close enough, $f(x) = \frac{1}{x}$ gets arbitrarily large, and you would want to "reduce the size of δ" so that $f(y) = \frac{1}{y}$ gets closer to $\frac{1}{x}$ so that we can reduce their distance down to ε).

Now, for the rigorous proof.

<u>f is continuous on $(0, \infty)$.</u> By the definition of ("plain") continuity, let $\varepsilon > 0$ and $x \in (0, \infty)$. This means that ε and x are fixed from the start. To complete the proof, we would like to find $\delta > 0$ such that if for any $y \in (0, \infty)$, if $0 < |x - y| < \delta$, then we have $|f(x) - f(y)| < \varepsilon$. Note that

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right|. \tag{43}$$

Since $|x - y| < \delta$ and x is fixed, it suffices for us to find an upper bound for $\left|\frac{1}{y}\right|$ or a lower bound for y (note that $y \in (0, \infty)$, so the absolute sign does not matter).
As we are free to pick whatever $\delta > 0$ that we like, we shall restrict our choice of δ to those in which $\delta \le \frac{x}{2}$ (this is perfectly legal since $x > 0$ so we do have some room to choose $\delta > 0$ from). Therefore, $0 < |x - y| < \delta$ implies that $x \neq y$ and $x - y > -\delta$ and $x - y < \delta \le \frac{x}{2}$. From the third inequality, we have $y > \frac{x}{2}$ and thus $\frac{1}{y} < \frac{2}{x}$. Back to (43), we have

$$|f(x) - f(y)| = \left|\frac{y - x}{xy}\right| < \frac{\delta}{x}\frac{2}{x} = \frac{2\delta}{x^2}. \tag{44}$$

Hence, if we choose $\delta \le \frac{x^2\varepsilon}{2}$, then $|f(x) - f(y)| < \varepsilon$.

Tallying our choice of δ, as long as we pick $\delta = \min\{\frac{x}{2}, \frac{\varepsilon x^2}{2}\}$, we are good to go! This concludes the proof.

<u>f is not uniformly continuous on $(0, \infty)$.</u>
Just because our choice of δ above depends on x, it does not directly prove that f is not uniformly continuous (there could be other choices of δ that do not depend on x). One would think of a "counterexample" here. However, what constitutes as a "counterexample"? Recall that f is uniformly continuous if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in (0, \infty), (0 < |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon).$$

By de Morgan's law, we can negate the statement to understand what is meant by "not uniformly continuous". Thus, we say that f is not uniformly continuous if

$$\begin{aligned}&\neg\left(\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in (0, \infty), (0 < |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon)\right)\\ &\exists \varepsilon > 0, \forall \delta > 0, \exists x, y \in (0, \infty), \neg(0 < |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon)\\ &\exists \varepsilon > 0, \forall \delta > 0, \exists x, y \in (0, \infty), 0 < |x - y| < \delta \text{ and } |f(x) - f(y)| \ge \varepsilon.\end{aligned} \tag{45}$$

Thus, to produce the counterexample, we have to submit a value of $\varepsilon > 0$. Let us consider $\varepsilon = 1$. (If you can understand the argument below, you will see that any choice of $\varepsilon > 0$ should work.) Then, let $\delta > 0$ be given. Now, pick a positive integer $N > 0$ sufficiently large such that $N > \sqrt{\delta}$, and consider $x = \frac{1}{N}$ and $y = \frac{1}{N+1}$ (such choices are legal since $N > 0$ and thus $x, y \in (0, \infty)$). Indeed, we can check that

$$0 < \left|\frac{1}{N} - \frac{1}{N+1}\right| = \left|\frac{1}{N(N+1)}\right| < \frac{1}{N^2} < \delta \tag{46}$$

and

$$|f(x) - f(y)| = |N - (N+1)| = 1 \ge 1. \tag{47}$$

Thus, f is not uniformly continuous on $(0, \infty)$.

Example 48. Let us fix a set $A = \{(x, y) \in \mathbb{R}^2 : x = y\}$. By using the sequential criterion for continuity, show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 1_A(x, y)$ is not continuous on $\mathbb{R}^2$.

Recall that f is continuous on $\mathbb{R}^2$ if it is continuous at every $(x, y) \in \mathbb{R}^2$. Thus, f is not continuous on $\mathbb{R}^2$ if there exists a point $(x, y) \in \mathbb{R}^n$ in which f is not continuous.

Recall that by sequential continuity, f is continuous at a given (x, y) if for every a sequence $(x_n, y_n) \subset \mathbb{R}^2$ converging to (x, y), we have $f(x_n, y_n)$ converging to $f(x, y)$. Thus, to prove that f is not continuous at some (x, y) by sequential continuity, it is sufficient to show that there is a sequence (x_n, y_n) converging to (x, y) but $f(x_n, y_n)$ does not converge to $f(x, y)$.

Pick say $(x, y) = (1, 1)$, and look at the sequence $\{\left(1 + \frac{1}{n}, 1 + \frac{1}{n}\right)\}_n$. One can see that $\left(1 + \frac{1}{n}, 1 + \frac{1}{n}\right)$ converges to $(1, 1)$ as n tends to ∞, but since $f\left(1 + \frac{1}{n}, 1 + \frac{1}{n}\right) = 1_A\left(1 + \frac{1}{n}, 1 + \frac{1}{n}\right) = 0$ for every n, then

$$\lim_{n\to\infty} f\left(1 + \frac{1}{n}, 1 + \frac{1}{n}\right) = \lim_{n\to\infty} 0 = 0 \neq f(1, 1).$$

Thus, f is not (sequentially) continuous on $\mathbb{R}^2$.

Exercises:

Exercise 5. Let n be any positive integer and $\mathbf{x}_0$ be an arbitrary point in $\mathbb{R}^n$. Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(\mathbf{x}) = 1_{\{\mathbf{x}\in\mathbb{R}^n : \mathbf{x}=\mathbf{x}_0\}}(x)$. By computing $U_N(f)$ and $L_N(f)$ for arbitrary N, use Definition 28 to show that f is integrable on $\mathbb{R}^n$, and

$$\int_{\mathbb{R}^n} f(\mathbf{x})\mathrm{d}V = 0. \tag{48}$$

Exercise 6. Let $X \subset \mathbb{R}^n$, $f : X \to \mathbb{R}^m$, and K be a positive real number. We say that f is K-Lipschitz if for every $\mathbf{x}, \mathbf{y} \in X$, we have

$$||f(\mathbf{x}) - f(\mathbf{y})|| \leq K||\mathbf{x} - \mathbf{y}||. \tag{49}$$

Prove that f is **uniformly** continuous on X.

Exercise 7. (Well-definedness of (i) of Theorem 31). Note that in (i), it was mentioned that f and g are integrable implies that $f + g$ is integrable. Strictly speaking, we can talk only about integrability of $f + g$ when it is a bounded function of bounded support. The purpose of this exercise is to prove that $f + g$ indeed possesses these properties.
(Notation for this Exercise: $\subseteq$ means "subset of or equals to". For the entire 32BH class in general, $\subset$ is the same as saying $\subseteq$.)

Thus, let us suppose that $f, g : \mathbb{R}^n \to \mathbb{R}$ are bounded functions with bounded support. *(Note: (ii) and (iii) are optional but challenging. Go for it if you are up for a challenge!)*

(i) Show that $f + g$ is bounded.

(ii) Prove the following set relation: For any two subsets A, B of $\mathbb{R}^n$,

$$A \subseteq B \implies \overline{A} \subseteq \overline{B}. \tag{50}$$

Here, the overline represents "closure" as defined in Discussion Supplement 1.

(iii) Prove the following set relation: For any two subsets A, B of $\mathbb{R}^n$,

$$\overline{A \cup B} = \overline{A} \cup \overline{B}. \tag{51}$$

(iv) By using (ii) and (iii), prove the following set relation:

$$\mathrm{supp}(f + g) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g). \tag{52}$$

(Recall that $\mathrm{supp}(h)$ here means support of a given function h, defined by $\mathrm{supp}(h) = \overline{\{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) \neq 0\}}$.)

(v) Use (iv) to prove that $f + g$ is a function with bounded support.

(vi) Is it true that

$$\mathrm{supp}(f + g) \supseteq \mathrm{supp}(f) \cup \mathrm{supp}(g)? \tag{53}$$

Note that if (53) is true, together with (52), we would have

$$\mathrm{supp}(f + g) = \mathrm{supp}(f) \cup \mathrm{supp}(g). \tag{54}$$

Exercise 8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Suppose that there exists a compact set A such that $\{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \neq 0\} \subset A$. Prove that f is integrable.

Partial Solutions/Hints:

- Exercise 5. One can show that $U_N(f) = \frac{1}{2^{nN}}$ and $L_N(f) = 0$. Thus, $U(f) = L(f) = 0$, and therefore, the required integral is 0.
- Exercise 6. Check directly by definition. For any given $\varepsilon > 0$, you should choose $\delta = \frac{\varepsilon}{K}$.
- Solution to Exercise 7 (ii). (Note that this is done purely by expanding the definition of closure and boundary of a set. It is possible to do this in a simpler way using properties of open and closed sets.) Assume that $A \subseteq B$. To show that $\overline{A} = A \cup \partial A \subseteq B \cup \partial B = \overline{B}$, we pick an element $x \in A \cup \partial A$.
  - If $x \in A$, then by $A \subseteq B$, $x \in B$, and thus $x \in B \cup \partial B = \overline{B}$ (since $B \subseteq B \cup \partial B$).
  - If $x \in \partial A$, then by definition of ∂A, for every $\varepsilon > 0$, $B_\varepsilon(x) \cap A \neq \emptyset$ and $B_\varepsilon(x) \cap A^c \neq \emptyset$.
  - Note that $B_\varepsilon(x) \cap B \supseteq B_\varepsilon(x) \cap A \neq \emptyset$ (since $A \subseteq B$ and we use the definition of ∂A in the second step) implies that $B_\varepsilon(x) \cap B \neq \emptyset$.
  - If $x \in B$ to begin with, then $x \in B \cup \partial B = \overline{B}$ immediately. Thus, it suffices to consider $x \notin B$, i.e, $x \in B^c$.
  - This implies that $B_\varepsilon(x) \cap B^c \supseteq B_\varepsilon(x) \cap \{x\} \supseteq \{x\} \cap \{x\} = \{x\} \neq \emptyset$. This is because if $x \in B^c$, the set B^c must contain at least the element x and thus $\{x\} \subseteq B^c$. Furthermore, the open ball $B_\varepsilon(x)$ must contain the element x if $\varepsilon > 0$, and thus $B_\varepsilon(x) \supseteq \{x\}$.
  - Combining $B_\varepsilon(x) \cap B \neq \emptyset$ and $B_\varepsilon(x) \cap B^c \neq \emptyset$, we have that $x \in \partial B$ and thus $x \in B \cup \partial B = \overline{B}$.
- Solution to Exercise 7 (iii).
  For $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$,
  - Note that

$$\begin{aligned} A \subseteq A \cup B &\xrightarrow[\text{apply (ii)}]{} \overline{A} \subseteq \overline{A \cup B}, \\ B \subseteq A \cup B &\xrightarrow[\text{apply (ii)}]{} \overline{B} \subseteq \overline{A \cup B}, \text{ and thus} \\ &\qquad \overline{A} \cup \overline{B} \subseteq \overline{A \cup B}. \end{aligned} \tag{55}$$

  For $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$,

  (a) Equivalently, we would like to show that

$$A \cup B \cup \partial(A \cup B) \subseteq A \cup B \cup \partial A \cup \partial B.$$

  Take an element $x \in A \cup B \cup \partial(A \cup B)$. If $x \in A$ or $x \in B$, then $x \in A \cup B$ and thus $x \in (A \cup B) \cup \partial A \cup \partial B$ and thus appears on the right hand side.

  (b) Thus, it suffices to show that if $x \in \partial(A \cup B)$, then $x \in A \cup B \cup \partial A \cup \partial B$.

  (c) Since $x \in \partial(A \cup B)$, by definition, for all $\varepsilon > 0$, we have $B_\varepsilon(x) \cap (A \cup B) \neq \emptyset$ and $B_\varepsilon(x) \cap (A \cup B)^c \neq \emptyset$.

  (d) If $x \in A$ or $x \in B$, then we are done (since $x \in A \cup B \ \cup \partial A \cup \partial B$, and thus right hand side is true). Thus, we will work with $x \in (A \cup B)^c$.

  (e) By (c), we have $B_\varepsilon(x) \cap (A \cup B) = (B_\varepsilon(x) \cap A) \cup (B_\varepsilon(x) \cap B) \neq \emptyset$. Since this set is not empty, either $B_\varepsilon(x) \cap A \neq \emptyset$ or $B_\varepsilon(x) \cap B \neq \emptyset$

  (f) Without loss of generality, suppose that $B_\varepsilon(x) \cap A \neq \emptyset$.

  (g) By (c), we have $B_\varepsilon(x) \cap (A \cup B)^c = B_\varepsilon(x) \cap A^c \cap B^c \neq \emptyset$. Since $B_\varepsilon(x) \cap A^c \supseteq B_\varepsilon(x) \cap A^c \cap B^c \neq \emptyset$, we have $B_\varepsilon(x) \cap A^c \neq \emptyset$.

  (h) Summarizing the two bullet points above, we have $B_\varepsilon(x) \cap A \neq \emptyset$ and $B_\varepsilon(x) \cap A^c \neq \emptyset$. Thus, $x \in \partial A$.

  (i) Suppose instead that in (e) and (f), we have $B_\varepsilon(x) \cap B \neq \emptyset$. One can repeat argument (g) and (h) to show that $x \in \partial B$.

  (j) In either cases, we have $x \in \partial A$ or $x \in \partial B$. Equivalently, $x \in \partial A \cup \partial B$. Thus, $x \in A \cup B \cup \partial A \cup \partial B$, as required.

- Exercise 7 (iv). First show that

$$\{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) + g(\mathbf{x}) \neq 0\} \subseteq \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \neq 0\} \cup \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \neq 0\}.$$

  Then, apply (ii), (iii), and the definition of supp.

- Exercise 7 (v). Recall the definition of **bounded support**, or in particular, what it means for a subset of $\mathbb{R}^n$ to be bounded.

- Exercise 7 (vi). Consider $f = 1_{[0,1]}$ and $g = -1_{[0,1]}$. Note that $f + g = 0$ (the zero function), and thus $\mathrm{supp}(f + g) = \emptyset$.

- Exercise 8. Apply the Heine Borel property of $\mathbb{R}^n$ to deduce that A is both closed and bounded. Recall that a set A is closed if $\overline{A} = A$ (see Discussion Supplement 1). Use the given condition and Exercise 7(ii) to show that the support of f is a subset of A, which is bounded. Now, apply Theorem 46.

3 Discussion 3

Summary for Lectures 6 - 7.

Before we begin, we start off with a review on two important theorems which will be used in this Discussion Supplement.

Theorem 49. (Heine-Borel Property on $\mathbb{R}^n$.) Let $K \subset \mathbb{R}^n$. We say that K is **compact** if and only if K is both closed and bounded.

Theorem 50. Let $K \subset \mathbb{R}^n$, K compact, and let $f : K \to \mathbb{R}^m$ be continuous. Then f is uniformly continuous on f.

We shall explicitly state one of the theorems used in Lecture 6 in the proof of Theorem 46 below.

Theorem 51. (Extreme Value Theorem on $\mathbb{R}^n$.) Let $K \subset \mathbb{R}^n$, K compact, and let $f : K \to \mathbb{R}^m$ be continuous. The range of f, given by

$$\mathrm{ran}(f) = \{f(x) : x \in K\} \tag{56}$$

is compact. In particular, the range of f is closed and bounded subset of $\mathbb{R}^m$. This further implies that f is a bounded function.

Recall the following result from the end of Lecture 5/start of Lecture 6:

Theorem 52. Let $f : \mathbb{R}^n \to \mathbb{R}$. If f is continuous with bounded support, then f is integrable.

(Note that for the integral to be well-defined, we require that f is also bounded. This follows from the fact that $f = 0$ outside of $\mathrm{supp}(f)$, and f is bounded on $\mathrm{supp}(f)$, which is closed and bounded, and thus f is bounded by the Extreme Value Theorem in $\mathbb{R}^n$.)

From the Challenge Problem Set 1, we also have the following theorem:

Theorem 53. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. Then, f is integrable if and only if f is continuous almost everywhere.

For specific details on what "almost everywhere" means, see Challenge Problem Set 1. Roughly speaking, the set of discontinuity is countable (that is, either finite, or countably infinite like $\mathbb{Q}$).

As we are forced to define integrals on $\mathbb{R}^n$, the first theorem is rarely useful due to the use of indicator functions to "cutoff" the function at certain points, which results in points of discontinuity. In contrast, since the indicator function only creates additional discontinuity at the "cutoff points", if the original function is continuous (almost everywhere), the resulting function obtained by multiplying with a "cutoff" function remains continuous almost everywhere. One should refer to the following example to observe the theorems in action!

Example 54. Let us consider the function $f : (0, \infty) \to (0, \infty)$ to be our (my) favorite function $f(x) = \frac{1}{x}$. Recall from Discussion Supplement 2 that on $(0, \infty)$, f is continuous but not uniformly continuous.

Consider $g(x) = f(x) \cdot 1_{1 \leq x \leq 3}(x)$. Thus, the support of g is given by $\overline{[1,3]} = [1,3]$, which is closed and bounded. Furthermore, g is continuous on $[1,3]$. By Extreme Value Theorem, g is bounded. In addition, note that:

- $g(1) = \frac{1}{1} = 1$ and $g(3) = \frac{1}{3}$, while $g(x) = 0$ for any $x < 1$ or $x > 3$. Thus, g is not continuous at 1 or 3
- g is continuous on $(-\infty, 1)$ or $(3, \infty)$ (since it is now the constant function 0 outside of $[1,3]$).
- g is also continuous on $(1,3)$, since $\frac{1}{x}$ is continuous on $(0, \infty)$ and thus on $(1,3)$.

With the above points, we know that the set of discontinuity is given by $\{1, 3\}$, is finite (and hence from Challenge Problem Set 1, it is of measure 0). Thus, g is continuous almost everywhere, and thus integrable by Theorem 53. Note that we will not be able to apply Theorem 46 since g has two discontinuous points, at 1 and 3.

Next, we consider $h(x) = f(x) \cdot 1_{0 \leq x < \infty}(x)$. Recall that the indicator function is really necessary as we have only defined integrals on $\mathbb{R}^n$. The support of h is given by $\overline{(0, \infty)} = [0, \infty)$, which is not bounded. It does not make sense to define integrals in terms of the definition introduced in this class.

Furthermore, we consider $k(x) = f(x) \cdot 1_{0 \leq x < K}(x)$ for some $K > 0$. The support of k is given by $\overline{(0, K)} = [0, K]$. Indeed, we now have that k has a bounded support. However, k is not bounded as it is defined on $(0, \infty)$, with $\operatorname{ran}(f) = (0, \infty)$ unbounded.

If we had considered $l(x) = f(x) \cdot 1_{K_1 \leq x < K_2}(x)$ for some $0 < K_1 < K_2$, this reduces to the first case that we have discussed (replace 1 by K_1 and 3 by K_2). Indeed, l is integrable on (K_1, K_2).

With all the theory of integrations in, we are ready to compute some integrals in $\mathbb{R}^n$.

Useful Properties:

Theorem 55. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two integrable functions. Then, we have the following properties.

(i) $f + g$ is integrable, and

$$\int_{\mathbb{R}^n} (f+g)(\mathbf{x})\mathrm{d}V = \int_{\mathbb{R}^n} f(\mathbf{x})\mathrm{d}V + \int_{\mathbb{R}^n} g(\mathbf{x})\mathrm{d}V. \tag{57}$$

(ii) If $\lambda \in \mathbb{R}$, then λf is integrable and

$$\int_{\mathbb{R}^n} \lambda f(\mathbf{x})\mathrm{d}V = \lambda \int_{\mathbb{R}^n} f(\mathbf{x})\mathrm{d}V. \tag{58}$$

(iii) Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ be two integrable functions. Then, $h : \mathbb{R}^{n+m} \to \mathbb{R}$ with $h(\mathbf{x}, \mathbf{y}) = f(\mathbf{x})g(\mathbf{y})$ defines an integrable function on $\mathbb{R}^{n+m}$, with

$$\int_{\mathbb{R}^{n+m}} h(\mathbf{x}, \mathbf{y})\mathrm{d}V = \int_{\mathbb{R}^n} f(\mathbf{x})\mathrm{d}V \int_{\mathbb{R}^m} g(\mathbf{y})\mathrm{d}V \tag{59}$$

Theorem 56. (Decomposition of Domains) Let K be a compact subset of $\mathbb{R}^n$ such that its boundary ∂D has volume 0. Furthermore, let $K = K_1 \cup K_2$, such that K_1 and K_2 are compact, and $K_1 \cap K_2$ has volume 0.
Let $f : \mathbb{R}^n \to K$ be a continuous function. Then f is integrable on K_1 and K_2 and

$$\int_K f(\mathbf{x})\mathrm{d}A = \int_{K_1} f(\mathbf{x})\mathrm{d}A + \int_{K_2} f(\mathbf{x})\mathrm{d}A \tag{60}$$

Theorem 57. (Fubini's Theorem.) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function that is bounded with bounded support. Then,

$$\int_{\mathbb{R}^n} f(\mathbf{x})\mathrm{d}V = \int_{-\infty}^{\infty} \cdots \left(\int_{-\infty}^{\infty} f(\mathbf{x}) dx_1 \right) \cdots dx_n \tag{61}$$

This means that the order of integration does not matter!

Definition 58. Let $f(\mathbf{x}, \mathbf{y}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a function. (Here, $\mathbf{x}$ denotes the first n variables and $\mathbf{y}$ denotes the next m variables.)
For a fixed $\mathbf{x}$, the function $f_{\mathbf{x}} : \mathbb{R}^m \to \mathbb{R}$ is defined by

$$f_{\mathbf{x}}(\mathbf{y}) := f(\mathbf{x}, \mathbf{y}). \tag{62}$$

Similarly, for a fixed $\mathbf{y}$, the function $f^{\mathbf{y}} : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$f^{\mathbf{y}}(\mathbf{x}) := f(\mathbf{x}, \mathbf{y}). \tag{63}$$

Thus, we have the following generalization of Fubini's Theorem:

Theorem 59. Let $f(\mathbf{x}, \mathbf{y}) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be an integrable function. Then, the functions $U(f_{\mathbf{x}}), L(f_{\mathbf{x}}), U(f^{\mathbf{y}})$, and $L(f^{\mathbf{y}})$ are integrable with

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} f\mathrm{d}V\mathrm{d}W = \int_{\mathbb{R}^n} U(f_{\mathbf{x}})\mathrm{d}V = \int_{\mathbb{R}^n} L(f_{\mathbf{x}})\mathrm{d}V = \int_{\mathbb{R}^m} U(f^{\mathbf{y}})\mathrm{d}W = \int_{\mathbb{R}^m} L(f^{\mathbf{y}})\mathrm{d}W. \tag{64}$$

With regards to Fubini's Theorem, here are some remarks:

- It is possible that $f(\mathbf{x},\mathbf{y})$ is integrable yet $f_{\mathbf{x}}(\mathbf{y})$ or $f^{\mathbf{y}}(\mathbf{x})$ are not. Take for instance, (assuming $(x,y)\in\mathbb{R}^2$)

$$f(x,y) := \begin{cases} 1 & \text{if } 0\le x<1 \text{ and } 0\le y\le 1 \\ 1 & \text{if } x=1, y\in\mathbb{Q}\cap[0,1] \\ 0 & \text{if otherwise.}\end{cases} \tag{65}$$

  This function is integrable since the set of discontinuities is a subset of $x=1$ with $0\le y\le 1$, a one-dimensional set over a two-dimensional region. From Challenge Problem Set 1, we know that the two-dimensional volume of such a one-dimensional set must be 0, and thus the set of discontinuity is of measure 0.
  (Note that if we are asking for the one-dimensional volume of $\mathbb{Q}\cap[0,1]$, it is undefined. However, if we are asking for the two-dimensional volume of $\{1\}\times\mathbb{Q}\cap[0,1]$, it is well-defined since the indicator function over this set is integrable in $\mathbb{R}^2$! Furthermore, the non-integrability of $1_{\mathbb{Q}\cap[0,1]}$ follows from the fact that this function is discontinuous on the entire $[0,1]$.)

  Yet, we know that $f(1,y)=1_{[0,1]\cap\mathbb{Q}}$ is not integrable in Discussion Supplement 2.

- U and L are there in case say f is integrable yet $f_{\mathbf{x}}$ or $f^{\mathbf{y}}$ are not. If they both are, then we reduce this to the case of the original Fubini's theorem, with a weakened assumption that f only has to be integrable.

- The notation in (61) might be misleading. In actual fact, for arbitrary sets $A\subset\mathbb{R}^n$ which are not boxes, it is possible for the limits of integration for the inner integrals to depend on the outer variables. For example,

$$\int_{y=0}^{y=2}\int_{x=y/2}^{x=1} e^{-x^2}\mathrm{d}x\mathrm{d}y. \tag{66}$$

  Notice that the limit of integration in $\int_{x=y/2}^{x=1} e^{-x^2}\mathrm{d}x$ indeed depends on y.

- In view of the above example (66), note that performing the inner integral first would be challenging as this would require us to know the general formula for the anti-derivative of e^{-x^2}. However, if we would want to seek help from Fubini's Theorem, we can't do a simple swap since

$$\int_{x=y/2}^{x=1}\int_{y=0}^{y=2} e^{-x^2}\mathrm{d}y\mathrm{d}x \tag{67}$$

  does not make sense as the output is a function that depends on the value of y. Instead, one should utilize Fubini's Theorem in the following way. First, we write (66) as

$$\int_{\mathbb{R}^2} e^{-x^2}1_B(x,y)\mathrm{d}A \tag{68}$$

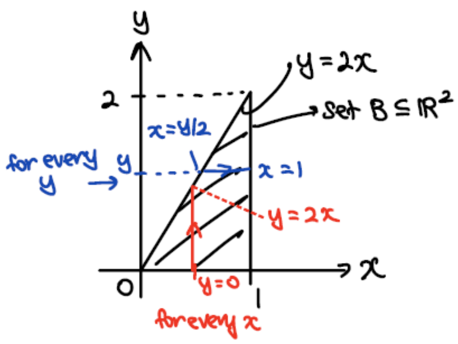
  where B is the set in which we are integrating over. To determine the set $B\subset\mathbb{R}^2$, we would have to draw a corresponding diagram corresponding to the region of integration in (66). Note that the integral should read: "as y increases from 0 to 2, the x value ranges from $y/2$ to 1." This allow us to draw the following diagram below. Looking at this shaded area labelled as B, we could view the diagram in another way: "as x increases from 0 to 1, the y value ranges from 0 to $2x$." Thus, (66) and (68) simplifies to

$$\int_{y=0}^{y=2}\int_{x=y/2}^{x=1} e^{-x^2}\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^2} e^{-x^2}1_B(x,y)\mathrm{d}A = \int_{x=1}^{x=0}\int_{y=0}^{y=2x} e^{-x^2}\mathrm{d}y\mathrm{d}x. \tag{69}$$

  Now, the rightmost expression in (69) can be evaluated directly as follows:

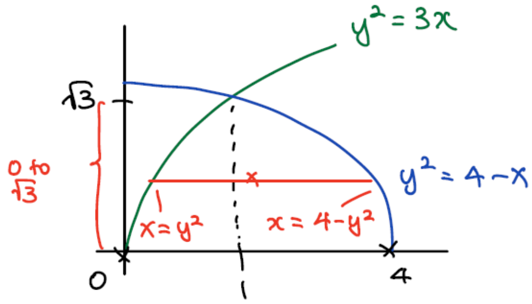
$$\int_{x=1}^{x=0}\int_{y=0}^{y=2x} e^{-x^2}\mathrm{d}y\mathrm{d}x = \int_{x=1}^{x=0} ye^{-x^2}|_{y=0}^{y=2x}\mathrm{d}x = \int_{x=1}^{x=0} 2xe^{-x^2}\mathrm{d}x = -e^{-x^2}|_{x=0}^{x=1} = 1-e^{-1}. \tag{70}$$

- Note that in the above example, the use of Fubini is justified in the sense of Theorem 59 **not** 57 for the same reason as our $\frac{1}{x}$ example, in which the indicator function creates discontinuity at the "cutoff" points. Instead, we say that f is integrable because it is bounded with bounded support, and continuous **almost everywhere** (by Theorem 53).



With that, let us look at another computational exercise that is slightly more involved.

Example 60. Evaluate the double integral $\int\int_D (x-y)\mathrm{d}A$ where D is the region above the x-axis bounded by $y^2 = 3x$ and $y^2 = 4 - x$.
Note that the curves $y^2 = 3x$ and $y^2 = 4 - x$ intersect at $(x, y) = (1, \sqrt{3})$. The region D is shown below.



In addition, note that it is convenient to conduct the inner integral in the x direction (since it is bounded above by the curve $x = 4 - y^2$ and below by $y^2/3$). If we did the inner integral in the y direction, we would have to split it up into two different cases; before $x = 1$, we have $0 \le y \le \sqrt{3x}$ and after $x = 1$ to $x = 4$, we have $0 \le y \le \sqrt{4-x}$. Furthermore, note that the appearance of square roots might introduce some possibly challenging integrals. Thus, the integral can thus be computed as follows:

$$\begin{aligned}
\int\int_D (x-y)\mathrm{d}A &= \int_0^{\sqrt{3}} \int_{y^2/3}^{4-y^2} (x-y)\mathrm{d}x\mathrm{d}y \\
&= \int_0^{\sqrt{3}} \left[\frac{1}{2}x^2 - yx\right]_{y^2/3}^{4-y^2} \mathrm{d}y \\
&= \int_0^{\sqrt{3}} \left(\frac{4}{9}y^4 + \frac{4}{3}y^3 - 4y^2 - 4y + 8\right) \mathrm{d}y \\
&= \left[\frac{4}{45}y^5 + \frac{1}{3}y^4 - \frac{4}{3}y^3 - 2y^2 + 8y\right]_0^{\sqrt{3}} \\
&= \frac{24}{5}\sqrt{3} - 3.
\end{aligned} \tag{71}$$

Note that the use of Fubini's is justified since $(x-y)1_D$ is a bounded function with bounded support, and continuous almost everywhere.

Exercises:

Exercise 9. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be the function defined as follows. For any $(x,y) \in [0,1] \times [0,1]$, we have

$$f(x,y) = \begin{cases} 1/y^2 & \text{if } 0 < x < y < 1 \\ -1/x^2 & \text{if } 0 < y < x < 1 \\ 0 & \text{if otherwise.} \end{cases} \tag{72}$$

(i) Compute $\int_0^1 \int_0^1 f(x,y)\mathrm{d}x\mathrm{d}y$.

(ii) Compute $\int_0^1 \int_0^1 f(x,y)\mathrm{d}y\mathrm{d}x$.

Exercise 10. Compute the following integral:

$$\int_{-1}^{1} \int_{y^{2/3}}^{(2-y)^2} \left(\frac{3}{2}\sqrt{x} - 2y\right) \mathrm{d}x\mathrm{d}y. \tag{73}$$

Exercise 11. Rewrite the integral

$$\int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} f(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x \tag{74}$$

as an iterated integral in the order of $\mathrm{d}x\mathrm{d}y\mathrm{d}z$.

Exercise 12. Find the region E for which the triple integral

$$\int\int\int_E (1 - x^2 - 2y^2 - 3z^2)\mathrm{d}V \tag{75}$$

is maximized. Briefly explain your answer.

Partial Solutions/Hints:

- Exercise 9. (i) yields 1 and (ii) yields -1. Fubini fails here because $f(x,y)$ is not integrable (because f is not bounded). See our favorite example on integrability of $1/x$ in Example 54 above and see if you can draw any similarities between this two-dimensional case and the one-dimensional case in that example!
  For the domain of integration, note that the line $y = x$ splits the square $[0,1] \times [0,1]$ into half along the diagonal, with $f(x,y) = 1/y^2$ above the diagonal and $-1/x^2$ below the diagonal. Thus, to compute say $\int_0^1 f(x,y)\mathrm{d}x$ for a given y, it makes more sense to split the integral as

$$\int_0^1 f(x,y)\mathrm{d}x = \int_0^y f(x,y)\mathrm{d}x + \int_y^1 f(x,y)\mathrm{d}x = \int_0^y \frac{1}{y^2}\mathrm{d}x + \int_y^1 -\frac{1}{x^2}\mathrm{d}x \tag{76}$$

  since for $\int_0^y \cdots \mathrm{d}x$, what we really mean is $0 < x < y$ and thus we have the case $f(x,y) = 1/y^2$.

- Exercise 10. Note that the domain of integration is really complicated so applying Fubini's to swap the order of integration might be challenging. Furthermore, there is no guarantee that the resulting integral is easier to solve. In fact, the integral by itself is a straightforward computation! One should get that the integral is given by $\frac{73}{3}$.
  Here, one should note that there will be a step involving $x^{3/2}$ evaluated at $x = y^{2/3}$. This expression yields $(y^{2/3})^{3/2} = |y|$, not y, in which this makes a difference when we integrate y from -1 to 1.

- Exercise 11. For every fixed $x \in (-1,1)$, we see that y is bounded from above by a straight line $y = 1$, and below by the parabola (cylinder if z is arbitrary) $y = x^2$. Furthermore, for every fixed (x,y), z is bounded from below by the plane $z = 0$, and from above by the plane $z = 1 - y$. The solution is given by

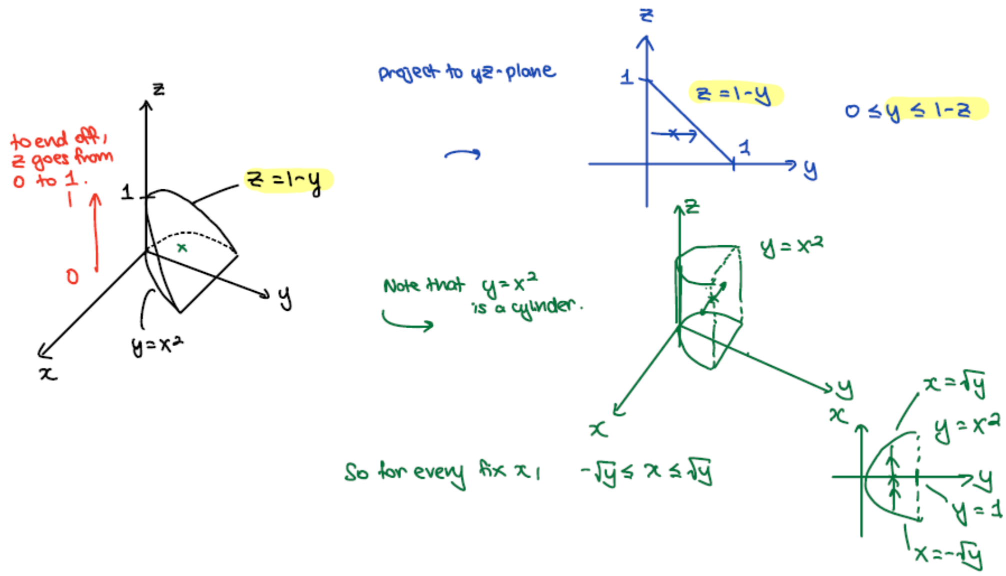
$$\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{77}$$

  See the attached diagram on the next page which serves as a visual aid for this exercise.

- Exercise 12. Note that $x^2 + 2y^2 + 3z^2$ is a non-negative function. Thus, $1 - x^2 - 2y^2 - 3z^2 \geq 0$ if $x^2 + 2y^2 + 3z^2 \leq 1$. Thus, we set E to be the ellipsoid

$$E = \{(x,y,z) \in \mathbb{R}^3 : x^2 + 2y^2 + 3z^2 \leq 1.\} \tag{78}$$

  The exterior region of E contributes to negative volume (since the integral is negative there), while if we consider any strict subset $F \subsetneq E$, we could have expanded F to E to increase the value of the triple integral as the integrand is non-negative in E. Thus, the integral is maximized with our choice of ellipsoid in (78).


to end off, z goes from 0 to 1.
z = 1-y
y = x²
project to yz-plane
z = 1-y
0 ≤ y ≤ 1-z
Note that y = x² is a cylinder.
y = x²
So for every fix x, -√y ≤ x ≤ √y
x = √y
y = x²
y = 1
x = -√y

4 Discussion 4

Summary for Lectures 8 - 10.

Polar Coordinates:

Proposition 61. Given a point (r, θ) in polar coordinates, we can compute the corresponding rectangular coordinates (x, y) by

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta). \end{cases} \tag{79}$$

Similarly, given a point (x, y) in rectangular coordinates, we can compute the corresponding polar coordinates (r, θ) (implicitly) by

$$\begin{cases} r & = \sqrt{x^2+y^2} \\ \tan(\theta) & = \frac{y}{x}. \end{cases} \tag{80}$$

Definition 62. A region R is **radially simple** if it is the region between graphs of two continuous functions $r_1(\theta)$ and $r_2(\theta)$ over a fixed interval of θ-values.
Mathematically, there exists $\alpha \leq \beta \in [0, 2\pi)$ and two continuous functions $r_1(\theta)$ and $r_2(\theta)$ defined on $[\alpha, \beta]$, such that

$$R = \{(r, \theta) : \alpha \leq \theta \leq \beta, r_1(\theta) \leq r \leq r_2(\theta)\}. \tag{81}$$

Theorem 63. If $f(x, y)$ is a continuous function on a (compact) radially simple region R, then

$$\int\int_R f(x,y)\mathrm{d}A = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos(\theta), r\sin(\theta)) r \mathrm{d}r \mathrm{d}\theta. \tag{82}$$

Spherical Coordinates:

Proposition 64. Given a point (ρ, θ, ϕ) in spherical coordinates, we can compute the corresponding Euclidean coordinates (x, y, z) by

$$\begin{cases} x = \rho\sin(\phi)\cos(\theta) \\ y = \rho\sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi). \end{cases} \tag{83}$$

Similarly, given a point (x, y, z) in Euclidean coordinates, we can compute the corresponding spherical coordinates (ρ, θ, ϕ) (implicitly) by

$$\begin{cases} \rho & = \sqrt{x^2+y^2+z^2} \\ \tan(\theta) & = \frac{y}{x} \\ \cos(\phi) & = \frac{z}{\rho}. \end{cases} \tag{84}$$

Definition 65. A solid region $R \subset \mathbb{R}^3$ is **centrally simple** if every ray from the origin intersects R in a single line segment such that the first endpoint lies on a surface $\rho = \rho_1(\theta, \phi)$ and the second endpoint lies on a surface $\rho = \rho_2(\theta, \phi)$. Mathematically, there exists $\theta_1 \leq \theta_2 \in [0, 2\pi)$, $\phi_1 \leq \phi_2 \in (-\pi/2, \pi/2)$, and two continuous functions $\rho_1(\theta, \phi)$ and $\rho_2(\theta, \phi)$ defined on the corresponding domain such that

$$R = \{(\rho, \theta, \phi) : \theta_1 \leq \theta \leq \theta_2, \phi_1 \leq \phi \leq \phi_2, \rho_1(\theta, \phi) \leq \rho \leq \rho_2(\theta, \phi)\}. \tag{85}$$

Theorem 66. If $f(x, y, z)$ is a continuous function on a (compact) centrally simple region R, then

$$\int\int\int_R f(x,y,z)\mathrm{d}V = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1(\theta,\phi)}^{\rho_2(\theta,\phi)} f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\ \rho^2\sin(\phi)\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta. \tag{86}$$

Cylindrical Coordinates:

Proposition 67. Given a point (r, θ, z) in cylindrical coordinates, we can compute the corresponding Euclidean coordinates (x, y, z) by

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z. \end{cases} \tag{87}$$

Similarly, given a point (x, y, z) in Euclidean coordinates, we can compute the corresponding cylindrical coordinates (r, θ) (implicitly) by

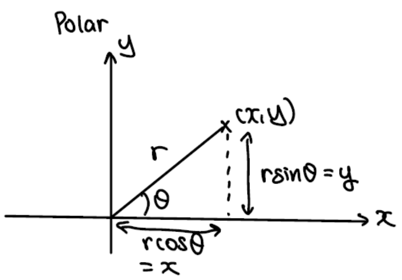
$$\begin{cases} r &= \sqrt{x^2+y^2} \\ \tan(\theta) &= \frac{y}{x} \\ z &= z. \end{cases} \tag{88}$$

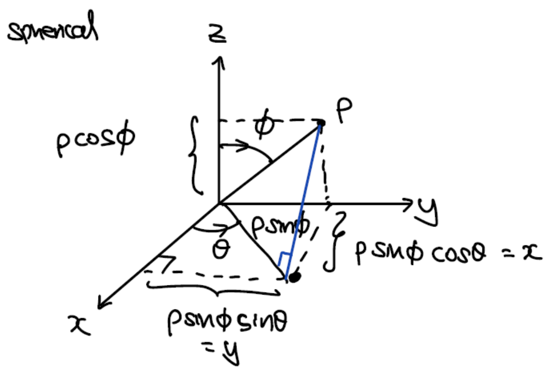
Theorem 68. If $f(x, y, z)$ is a continuous function on a (compact) region C such that for any given $z = z_0$ in C, $C|_{z=z_0}$ is a radially simple region, then

$$\int\int\int_C f(x,y,z)\mathrm{d}V = \int_a^b\int_{\alpha(z)}^{\beta(z)}\int_{r_1(\theta,z)}^{r_2(\theta,z)} f(r\cos(\theta), r\sin(\theta), z)r\mathrm{d}r\mathrm{d}\theta dz. \tag{89}$$

Here, the z-coordinates for C goes from a to b (with $a \leq b$).

Diagramatic Aid:





General change of variables formula:

Theorem 69. Let $K \subset \mathbb{R}^n$ be a compact set with $\mathrm{vol}_n(\partial K) = 0$. Let $U \subset \mathbb{R}^n$ be an open set containing K and let

$$\Phi : U \to \mathbb{R}^n \tag{90}$$

be a map such that

1. Φ is a C^1 mapping,
2. Φ is injective on $\mathrm{int}(K)$, and
3. $\det(D\Phi) \neq 0$ on $\mathrm{int}(K)$.

Then, if $f : \Phi(K) \to \mathbb{R}$ is a continuous function, then

$$\int_{\Phi(K)} f(\mathbf{x}) \mathrm{d}V(\mathbf{x}) = \int_K (f \circ \Phi)(\mathbf{y}) \cdot |\det(D\Phi)| \mathrm{d}V(\mathbf{y}). \tag{91}$$

There are a lot of technical terms introduced in Theorem 69. We shall introduce them in a sequential manner, providing examples if possible at each step to aid the reader in internalizing the definitions/concepts.

Definition 70. A subset $U \subset \mathbb{R}^n$ is **open** if for every $x \in U$, there exists $r > 0$ such that $B_r(x) \subset U$.

Definition 71. The **interior** of $X \subset \mathbb{R}^n$ is defined as

$$\mathrm{int}(X) = X^\circ := X \setminus \partial X = \{\mathbf{x} \in X : \mathbf{x} \notin \partial X\}. \tag{92}$$

Proposition 72. For any subset $X \subset \mathbb{R}^n$, X° is open.

Example 73. Consider $X = [1, 3) \subset \mathbb{R}$. Recall that $\partial X = \{1, 3\}$. With that, we can compute $X^\circ = [1, 3) \setminus \{1, 3\} = (1, 3)$. Indeed, one can also see that X°, an open interval, is open. (Note that just because we call $(1, 3)$ an open interval does not automatically implies that $(1, 3)$ is an open set - this requires proof!)

Definition 74. A function $f : A \subset \mathbb{R}^m \to \mathbb{R}^n$ is **differentiable** at some $\mathbf{x}_0 \in \text{int}(A)$ if there exists a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{\mathbf{h}\to 0} \frac{||f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - T(\mathbf{h})||}{||\mathbf{h}||} = 0. \tag{93}$$

If f is indeed differentiable at $\mathbf{x}_0$, then we say that the **derivative** of f at $\mathbf{x}_0$ is given by

$$Df(\mathbf{x}_0) := T. \tag{94}$$

Equality above can be understood as equality of linear transformations, in which for all $\mathbf{h} \in \mathbb{R}^m$, we have $Df(\mathbf{x}_0)(\mathbf{h}) = T(\mathbf{h})$.[2]

Definition 75. Since the co-domain of the function $f : A \subset \mathbb{R}^m \to \mathbb{R}^n$ is in $\mathbb{R}^n$, this implies that we can write $f(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^m$ as

$$f(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix} \tag{95}$$

whereby $f_i : A \subset \mathbb{R}^m \to \mathbb{R}$ for $i \in \{1, ..., n\}$ and $\mathbf{x} = (x_1, ..., x_m)$. The **Jacobian** matrix of f at $\mathbf{x}_0$ is thus given by

$$[J_f(\mathbf{x}_0)] = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix} (\mathbf{x}_0). \tag{96}$$

(The right hand side of (96) is to be understood as the matrix evaluated at $\mathbf{x}_0$.)

Theorem 76. Let $f : A \subset \mathbb{R}^m \to \mathbb{R}^n$. If f is differentiable at $\mathbf{x}_0 \in \text{int}(A)$, then all ($1 \leq i \leq m$ and $1 \leq j \leq n$) the (first order) partial derivatives $\frac{\partial}{\partial x_i} f_j(\mathbf{x}_0)$ exists and the standard matrix[a] of $Df(\mathbf{x}_0)$ is $[J_f(\mathbf{x}_0)]$. Mathematically, we have

$$Df(\mathbf{x}_0)(\mathbf{h}) = [J_f(\mathbf{x}_0)]\mathbf{h}. \tag{97}$$

[a] Standard matrix here refers to the representation of the matrix in the standard basis in $\mathbb{R}^m$. A definition is given in Theorem 87. Such a concept will be covered in depth either in 33A or 115A.

Example 77. Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$f(x, y) = \begin{pmatrix} xy \\ \sin(x + y) \\ x^2 - y^2. \end{pmatrix} \tag{98}$$

One can compute the corresponding Jacobian matrix at any arbitrary point $(x, y) \in \mathbb{R}^2$ as follows:

$$[J_f(x, y)] = \begin{pmatrix} \frac{\partial}{\partial x}(xy) & \frac{\partial}{\partial y}(xy) \\ \frac{\partial}{\partial x}(\sin(x + y)) & \frac{\partial}{\partial y}(\sin(x + y)) \\ \frac{\partial}{\partial x}(x^2 - y^2) & \frac{\partial}{\partial y}(x^2 - y^2) \end{pmatrix} = \begin{pmatrix} y & x \\ \cos(x + y) & \cos(x + y) \\ 2x & -2y \end{pmatrix}. \tag{99}$$

[2] If you do not know what a linear transformation is because you do not have credits for 33A or did not take 32AH, just think of a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ as a $n \times m$ matrix.

Definition 78. A function $f : A \subset \mathbb{R}^m \to \mathbb{R}^n$ is a C^1 **mapping** if it is differentiable on A and all its partial derivatives are continuous on A.

(In the language of single variable calculus, for any $f : \mathbb{R} \to \mathbb{R}$, we say that f is C^1 if it is continuously differentiable.)

Proposition 79. If a function $f : A \subset \mathbb{R}^m \to \mathbb{R}^n$ is differentiable on A if all its partial derivatives are continuous on A.

The above proposition implies that it suffices to compute the partial derivatives (or preferably the Jacobian matrix directly) and see if they are continuous to prove that f is indeed differentiable on A, and thus we can define the derivative of f as in (94) and apply Theorem 76 directly (which we can use to Jacobian matrix previously computed as the derivative directly).

Definition 80. A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is **injective** (one-to-one) on a subset $A \subset \mathbb{R}^n$ if for all $\mathbf{u}, \mathbf{v} \in A$, $f(\mathbf{u}) = f(\mathbf{v}) \implies \mathbf{u} = \mathbf{v}$.

(Equivalently, for all $\mathbf{u}, \mathbf{v} \in A, \mathbf{u} \neq \mathbf{v} \implies f(\mathbf{u}) \neq f(\mathbf{v})$. Intuitively, this means that "different inputs must be mapped to different outputs".)

Definition 81. A function $f : X \to Y$ is **invertible** if there exists a function $g : Y \to X$ such that for all $\mathbf{x} \in X$, $g(f(\mathbf{x})) = x$ and for all $\mathbf{y} \in Y$, we have $f(g(\mathbf{y})) = \mathbf{y}$.

Proposition 82. Let $X, Y \subset \mathbb{R}^n$ and $f : X \to Y$. If a function f is invertible, then f is injective on X.

Example 83. Consider $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ given by $f(x) = x^2$.

- f is injective on $\mathbf{R} \cup \{0\}$. We can check this by definition. For any $x, y \in \mathbf{R} \cup \{0\}$, $f(x) = f(y)$ implies that $x^2 = y^2$. Since both $x, y \geq 0$, we can take the positive root on both sides to obtain $x = y$, as required to prove that f is indeed injective on this domain.

- f is not injective on $\mathbb{R}$. We can prove this by providing a counterexample, since

$$\begin{aligned} &\neg(\forall x, y \in \mathbb{R}, f(x) = f(y) \implies x = y) \\ &= (\exists x, y \in \mathbb{R}, f(x) = f(y) \text{ and } x \neq y.) \end{aligned}$$

  Thus, a counterexample here would be $x = 1$, $y = -1$, and $f(x) = 1^2 = 1 = (-1)^2 = f(y)$.

- Since f is not injective on $\mathbb{R}$, then f is not invertible by the contrapositive of Proposition 82.

- Consider $f_1 : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ given by $f_1(x) = f(x)$ for all $x \in \mathbb{R}^+ \cup \{0\}$. We claim that f_1 is invertible. Consider the "inverse" function $g : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ given by $g(x) = \sqrt{x}$. One can check that $f(g(x)) = (\sqrt{x})^2 = x$ and $g(f(x)) = \sqrt{x^2} = x$ (since $x \geq 0$). By Proposition 82, f_1 is injective on $\mathbb{R}^+ \cup \{0\}$, which is what we have checked in the first bullet point.

Next, we look at some applications of the general change of variables formula as in Theorem 69.

Theorem 84. (The other direction.) Let $K \subset \mathbb{R}^n$ be a compact set with $\mathrm{vol}_n(\partial K) = 0$. Let $U \subset \mathbb{R}^n$ be an open set containing K and let

$$\Phi : U \to \mathbb{R}^n \tag{100}$$

be a map such that

1. Φ is a C^1 mapping,
2. Φ is injective on $\mathrm{int}(K)$, and
3. $\det(D\Phi) \neq 0$ on $\mathrm{int}(K)$.

Then, if $f : \Phi(K) \to \mathbb{R}$ is a continuous function, then

$$\int_K (f \circ \Phi)(\mathbf{x}) \mathrm{d}V(\mathbf{x}) = \int_{\Phi(K)} f(\mathbf{y}) \cdot \frac{1}{|\det(D\Phi)|} \mathrm{d}V(\mathbf{y}). \tag{101}$$

This can be understood by comparing say $\Phi(\mathbf{x}) = \mathbf{y}$ for this case as compared to $\mathbf{x} = \Phi(\mathbf{y})$. To deduce this from Theorem 69, since Φ is injective on $\mathrm{int}(K)$, $\Phi : \mathrm{int}(K) \to \Phi(\mathrm{int}(K))$ is invertible (bijective) and thus we can invert $\Phi(\mathbf{x}) = \mathbf{y}$ to $\mathbf{x} = \Phi^{-1}(\mathbf{y})$ and substitute this in the corresponding formula. Furthermore, we have also used the following fact:

Theorem 85. If $G = F^{-1}$ and $\det(DF) \neq 0$, then $\det(DG) = \frac{1}{\det(DF)}$.

In addition, we can also look at the case for linear maps/transformations below.

Definition 86. A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear map** if

$$T\left(\sum_{i=1}^{k} \alpha_i \mathbf{v}_i\right) = \sum_{i=1}^{k} \alpha_i T(\mathbf{v}_i) \tag{102}$$

for all $k \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$, and all vectors $\mathbf{v}_i \in \mathbb{R}^n$.

Theorem 87. A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if there is a matrix $A \in M_{m\times n}(\mathbb{R})$ such that

$$T(\mathbf{x}) = A\mathbf{x}. \tag{103}$$

We call A the **standard matrix** of T.

Notation: $M_{m\times n}(\mathbb{R})$ refers to the set of $m \times n$ matrices with real entries.

Definition 88. A matrix $A \in M_{m\times n}(\mathbb{R})$ is **invertible** if there exists a matrix $B \in M_{m\times n}(\mathbb{R})$ such that $AB = BA = I_n$. Here, B is known as the **inverse** of A.

Corollary 89. A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if its standard matrix A is invertible.

Theorem 90. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with standard matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem 91. (Computing determinants using cofactor expansion.) Let $A \in M_{n\times n}(\mathbb{R})$, and let $A_{ij} \in M_{(n-1)\times(n-1)}(\mathbb{R})$ denote the matrix obtained by deleting row i and column j from A. Then, we have

$$\det(A) = (-1)^{i+1}a_{i,1}\det(A_{i1}) + ... + (-1)^{i+n}a_{i,n}\det(A_{in}). \tag{104}$$

Here, we note that the equation above is true for all valid choices of i, and thus is independent of the choice of i (that is, the row in which you plan to expand the theorem about to compute the determinant).

For instance, if $A \in M_{2\times 2}(\mathbb{R})$, we have

$$\begin{aligned} \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (-1)^{1+1}a_{11}\det(A_{11}) + (-1)^{1+2}a_{12}\det(A_{12}) \\ &= a\det(d) - b\det(c) = ad - bc. \end{aligned} \tag{105}$$

Theorem 92. The volume of a parallelpiped spanned by $\mathbf{v}_1, ..., \mathbf{v}_n$ in $\mathbb{R}^n$ is given by

$$\mathrm{vol}_n(D) = \left| \det\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} \right|. \tag{106}$$

Note that $\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{pmatrix}$ is an $n \times n$ matrix with the first row given by the entire vector $\mathbf{v}_1$, etc.

Last but not least, we arrive at what we want to conclude for linear transformations:

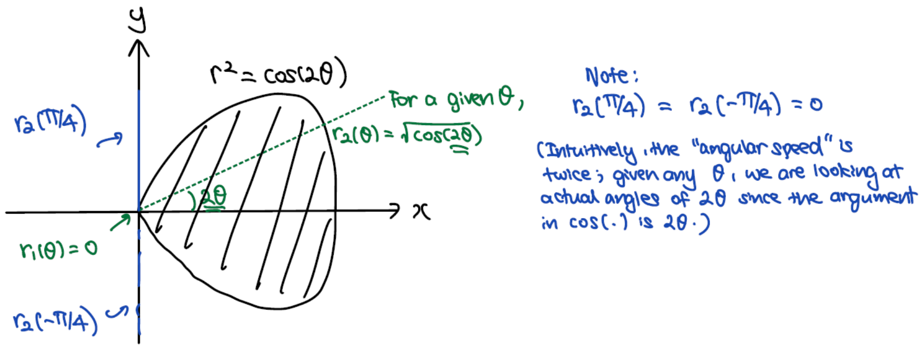
Theorem 93. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation, and let $f : \mathbb{R}^n \to \mathbb{R}$ be an **integrable** function. Then $f \circ T : \mathbb{R}^n \to \mathbb{R}$ is integrable and

$$\int_{\mathbb{R}^n} f(\mathbf{x})\mathrm{d}V(\mathbf{x}) = \int_{\mathbb{R}^n} (f\circ T)(\mathbf{y})|\det T|\mathbf{d}V(\mathbf{y}). \tag{107}$$

One can also show that $|\det T|$ is a constant that does not depends on $\mathbf{y}$. This will be the content for Exercise 17 below. Furthermore, $\det(T)$ here is understood as the determinant of the **standard matrix** of T.

With all the tools present, we shall look at some of the computational examples.

Example 94. (Area of a lemniscate). Consider the curve given by the equation $r^2 = \cos(2\theta)$. Compute the area of the right lobe of the curve.



Let A denote the region corresponding to the right loop of the lemniscate in rectangular coordinates (x, y). By switching to the polar coordinates, we can obtain a corresponding radially simple region B given by

$$B = \{(r, \theta) : -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \text{ and } 0 < r < \sqrt{\cos(2\theta)}.\} \tag{108}$$

Thus, the corresponding area is given by

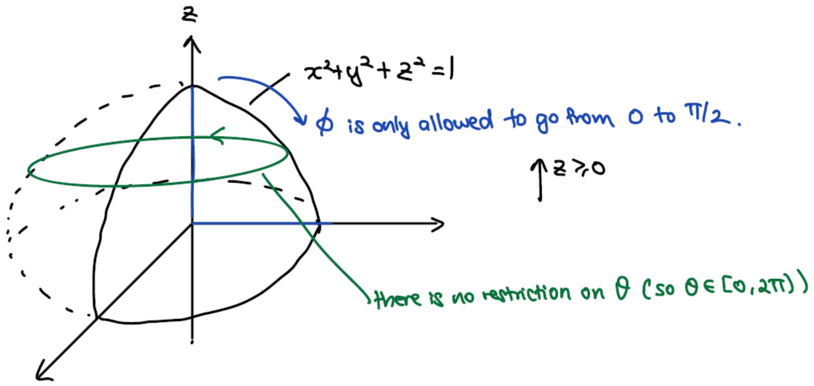
$$\begin{aligned} \int\int_A 1 \mathrm{d}A &= \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos(2\theta)}} 1 \cdot r \mathrm{d}r \mathrm{d}\theta \\ &= \int_{-\pi/4}^{\pi/4} \frac{\cos(2\theta)}{2} \mathrm{d}\theta = \frac{1}{2}. \end{aligned} \tag{109}$$

(We can apply the change of variables formula as in Theorem 63 since 1 here is continuous on A and the radially simple region is compact.)

Example 95. (A practice question on spherical coordinates.) Let the region $R \subset \mathbb{R}^3$ be given in Euclidean coordinates by

$$R = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, z \geq 0\}. \tag{110}$$

Compute $\int\int\int_A z \mathrm{d}x\mathrm{d}y\mathrm{d}z$.



From the "spherical symmetry" hinted by the region R, we will thus consider the use of spherical coordinates. The corresponding region in spherical coordinates is given by

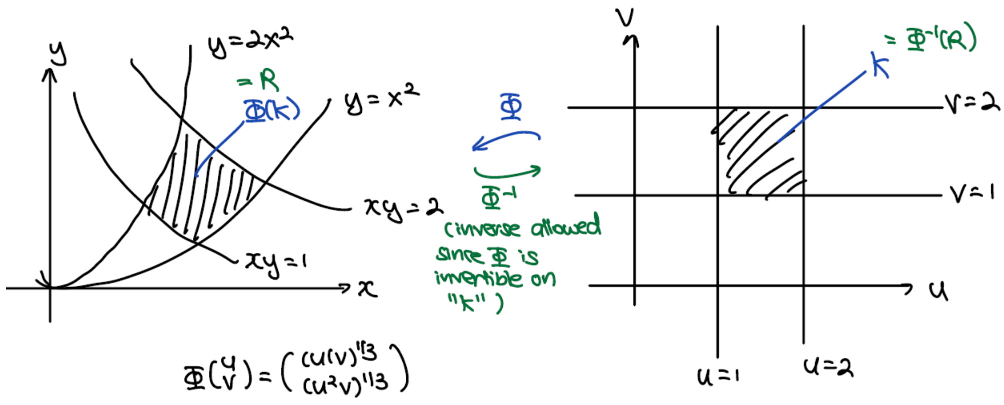
$$S = \{(r, \theta, \phi) \in (0, \infty) \times [0, 2\pi) \times (-\pi/2, \pi/2) : r \leq 1 \text{ and } \phi \geq 0\}. \tag{111}$$

Here, we note that $\phi \geq 0$ corresponds to the fact that the z-coordinates in R are non-negative. By the change of variables formula as in Theorem 66, we have

$$\begin{aligned}
\int\int\int_A z \mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} (r \sin(\phi)) \cdot r^2 \cos(\phi) \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}r \\
&= \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} r^3 \sin(\phi) \cos(\phi) \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}r \\
&= \left(\int_0^1 r^3 \mathrm{d}r\right) \left(\int_0^{2\pi} \mathrm{d}\theta\right) \left(\int_0^{\pi/2} \sin(\phi) \cos(\phi) \mathrm{d}\phi\right) \\
&= \left(\int_0^1 r^3 \mathrm{d}r\right) \left(\int_0^{2\pi} \mathrm{d}\theta\right) \left(\int_0^{\pi/2} \frac{\sin(2\phi)}{2} \mathrm{d}\phi\right) \\
&= \frac{1}{4} \cdot (2\pi) \cdot \frac{1}{2} = \frac{\pi}{4}.
\end{aligned} \tag{112}$$

Example 96. (Change of variables I.) Find the area of the region $R \subset \mathbb{R}^2$ given by

$$R = \{1 \leq xy \leq 2 \text{ and } x^2 \leq y \leq 2x^2\}. \tag{113}$$



Note that $x \neq 0$ else $1 \leq xy \leq 2$ does not hold. This implies that we can divide by x^2 in the second inequality to obtain

$$R = \{1 \leq xy \leq 2 \text{ and } 1 \leq \frac{y}{x^2} \leq 2\}. \tag{114}$$

An instructive "substitution" would be to set $u = xy$ and $v = \frac{y}{x^2}$. To be mathematically rigorous, we shall attempt to apply Theorem 69. This implies that we need to find the corresponding mapping Φ such that

$$\begin{aligned} \int_{\Phi(K)} f(\mathbf{x}) \mathrm{d}V(\mathbf{x}) &= \int_K (f \circ \Phi)(\mathbf{y}) \cdot |\det(D\Phi)| \mathrm{d}V(\mathbf{y}) \\ \int_R 1_R(\mathbf{x}) \mathrm{d}V(\mathbf{x}) &= \int_{\Phi^{-1}(R)} 1_R(\Phi(\mathbf{y})) \cdot |\det(D\Phi)| \mathrm{d}V(\mathbf{y}) \end{aligned} \tag{115}$$

Here, it is understood that $\mathbf{x} = \Phi(\mathbf{y})$. Thus, if $\mathbf{x} = (x, y)$ and $\mathbf{y} = (u, v)$, this implies that we want to find Φ such that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi \begin{pmatrix} u \\ v \end{pmatrix} \tag{116}$$

which in otherwords, using the "substitution", we have to find explicit forms of x and y in terms of u and v. Solving $u = xy$ and $v = y/x^3$, we obtain

$$\begin{cases} x = (u/v)^{\frac{1}{3}} \\ y = (u^2 v)^{\frac{1}{3}} \end{cases} \tag{117}$$

and thus

$$\Phi \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (u/v)^{\frac{1}{3}} \\ (u^2 v)^{\frac{1}{3}} \end{pmatrix}. \tag{118}$$

(Note that the x and y's are indeed solvable since $v = y/x^2$ is defined as $x \neq 0$ as explained above.) Next, we shall proceed to verify the hypotheses of Theorem 69.

Φ is injective on $\operatorname{int}(\Phi^{-1}(R))$. We show that Φ is injective on $\operatorname{int}(\Phi^{-1}(R)) = \{(u, v) : 1 < u < 2 \text{ and } 1 < v < 2\}$. Note that since Φ is obtained by solving x and y in terms of u and v, we effectively have that if we define

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Psi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ y/x^2 \end{pmatrix}, \tag{119}$$

with $\Psi\Phi\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}u\\v\end{pmatrix}$ and $\Phi\Psi\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$. (Basically, by obtaining Φ, we inverted Ψ in the corresponding domain.) Since $\Phi:\mathrm{int}(\Phi^{-1}(R))\to\Phi(\mathrm{int}(\Phi^{-1}(R)))$ is invertible, then it is injective on $\mathrm{int}(\Phi^{-1}(R))$ (see Proposition 82).

Φ is a C^1 mapping and $\det(D\Phi)\neq 0$ on $\mathrm{int}(\Phi^{-1}(R))$. In view of Proposition 79, we compute

$$[J_\Phi(\mathbf{y})]=\begin{pmatrix}\frac{\partial(u/v)^{1/3}}{\partial u} & \frac{\partial(u/v)^{1/3}}{\partial v}\\ \frac{\partial(u^2v)^{1/3}}{\partial u} & \frac{\partial(u^2v)^{1/3}}{\partial v}\end{pmatrix}=\begin{pmatrix}\frac{1}{3u^{2/3}v^{1/3}} & -\frac{u^{1/3}}{3v^{4/3}}\\ \frac{2v^{1/3}}{3u^{1/3}} & \frac{u^{2/3}}{3v^{2/3}}\end{pmatrix} \tag{120}$$

which we can see that the partial derivatives are continuous since $v,u\neq 0$ (ie $u=0$, then $xy=0$ which is outside of the range; alternatively, see the corresponding domain $\mathrm{int}(\Phi^{-1}(R))$). Furthermore, $|\det(D\Phi(\mathbf{y})|=|\det(J_\Phi(\mathbf{y}))|=\frac{1}{3v}\neq 0$. Thus, Φ is a C^1 mapping and $\det(D\Phi)\neq 0$.

Furthermore, the function 1 is clearly continuous on R. Thus, by the general change of variables formula, we have

$$\begin{aligned}\int_R \mathrm{d}A(\mathbf{x}) &= \int_{\Phi^{-1}(R)}\Phi(\mathbf{y})\cdot|\det(D\Phi(\mathbf{y}))|\mathrm{d}A(\mathbf{y})\\ &=\int_1^2\int_1^2\frac{1}{3v}\mathrm{d}u\mathrm{d}v\\ &=(2-1)\cdot\frac{1}{3}\cdot\ln(2)=\frac{\ln(2)}{3}.\end{aligned} \tag{121}$$

Example 97. (Change of variables II.) Find the area of the region $R \subset \mathbb{R}^2$ given by

$$R = \{1 \le xy \le 2 \text{ and } x^2 \le y \le 2x^2\}. \tag{122}$$

(Same question as Example 96.)
Recall that an instructive "substitution" would be to set $u = xy$ and $v = \frac{y}{x^2}$. However, instead of solving x and y for u and v, we can instead apply the change of variables formula in the reverse direction, as indicated in Theorem 84. In the language of Theorem 84, we have

$$\int_R (f \circ \Phi)(\mathbf{x}) \mathrm{d}V(\mathbf{x}) = \int_{\Phi^{-1}(R)} f(\mathbf{y}) \cdot \frac{1}{|\det(D\Phi(\mathbf{y}))|} \mathrm{d}V(\mathbf{y}). \tag{123}$$

This time round, if $\mathbf{x} = (x, y)$ and $\mathbf{y} = \Phi(\mathbf{x})$ with $\mathbf{y} = (u, v)$, it is clear that

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ \frac{y}{x^2} \end{pmatrix}. \tag{124}$$

Although one would have to show the injectivity of Φ by solving for x and y in terms of u and v as in the previous example (to construct the inverse and thus is injective), the advantage of this method is that the computation of $D\Phi$ is actually simpler and shorter. For instance, we have

$$[J_\Phi(\mathbf{x})] = \begin{pmatrix} \frac{\partial(xy)}{\partial x} & \frac{\partial(xy)}{\partial y} \\ \frac{\partial(y/x^2)}{\partial x} & \frac{\partial(y/x^2)}{\partial y} \end{pmatrix} = \begin{pmatrix} y & x \\ -2y/x^3 & 1/x^2 \end{pmatrix} \tag{125}$$

and thus

$$|\det(D\Phi(\mathbf{x}))| = 3y/x^2 = 3v \tag{126}$$

(Here, we write the output as a function of $\mathbf{y}$ since technically speaking, in the formula above, we would require $|\det(D\Phi(\mathbf{y})|$, that is, the Jacobian expressed in the "new" coordinate system that we are working with.) Of course, one can conclude from above that Φ is differentiable and $\det(D\Phi) \neq 0$ (since $v \neq 0$).

Using the formula as in (123) and setting $f = 1_R$ which is continuous on R, we have

$$\begin{aligned} \int_R (1_R \circ \Phi)(\mathbf{x}) \mathrm{d}V(\mathbf{x}) &= \int_{\Phi^{-1}(R)} 1_R(\mathbf{y}) \cdot \frac{1}{|\det(D\Phi)|} \mathrm{d}V(\mathbf{y}) \\ &= \int_1^2 \int_1^2 1 \cdot \frac{1}{3v} \mathrm{d}v \mathrm{d}u = \frac{\ln(2)}{3}. \end{aligned} \tag{127}$$

Note that the final area as computed using both methods coincide, as expected!

Exercises:

Exercise 13. Let $f(x,y) = \begin{cases} x^2+y^2 & \text{if } x^2+y^2 \le R^2 \\ 0 & \text{if } x^2+y^2 > R^2. \end{cases}$ Compute $\int_{\mathbb{R}^2} f(x,y)\mathrm{d}A$.

Exercise 14. Compute the following integral:

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \sqrt{x^2+y^2+z^2}\mathrm{d}z\mathrm{d}y\mathrm{d}x. \tag{128}$$

Exercise 15. (Basic Topology on $\mathbb{R}^n$). Recall the definition of open sets in $\mathbb{R}^n$ as in 70. Furthermore, recall the definition of closed sets in $\mathbb{R}^n$, namely, a subset $X \subset \mathbb{R}^n$ is closed if $\overline{X} = X$ or for any sequence $x_n \in A \to x \in \mathbb{R}^n$, then $x \in A$. In this exercise, we shall prove some of the basic properties of open and closed sets, and apply them to interiors and boundaries of sets.
You may assume without proof that the following properties hold:

- Given any open set $A \subset \mathbb{R}^n$, $\mathbb{R}^n \setminus A$ is closed.
- Conversely, given any closed set $A \subset \mathbb{R}^n$, $\mathbb{R}^n \setminus A$ is open.
- The closure $\overline{A}$ of any subset A of $\mathbb{R}^n$ is closed.

Prove the following:

(i) Given any two open subsets A and B of $\mathbb{R}^n$, prove that $A \cup B$ and $A \cap B$ are open.

(ii) Given any two closed subsets A and B of $\mathbb{R}^n$, prove that $A \cup B$ and $A \cap B$ are closed.

(iii) Using some of the properties above, prove that for any $A \subset \mathbb{R}^n$, ∂A is closed.

(iv) Using some of the properties above, prove that for any $A \subset \mathbb{R}^n$, A° is open.

Exercise 16. By using the general change of variables formula, compute the volume of an ellipsoid, a solid bounded by the surface (with $a, b, c > 0$)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{129}$$

Exercise 17. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Prove the following:

(i) $|\det(DT)| = |\det(T)|$,[a] and

(ii) $|\det(T)|$ is a constant depending only on the given T.

[a] det here is understood as the determinant of the standard matrix for T.

Partial Solutions/Hints:

- Exercise 13. Note that the support of the function is compact and that the function is continuous. The support here is basically a ball of radius R centered at the origin. With that, we can apply the change of variable formula for polar coordinates to obtain

$$\int_{\mathbb{R}^2} f(x,y)\mathrm{d}A = \int_0^{2\pi}\int_0^R f(r\cos\theta, r\sin\theta)r\mathrm{d}r\mathrm{d}\theta = ... = \frac{\pi R^4}{2}. \tag{130}$$

- Exercise 14. Use spherical coordinates! Upon figuring out the right domain, you should have

$$... = \int_0^{\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^3\sin(\phi)\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta = \pi\left(1 - \frac{1}{\sqrt{2}}\right). \tag{131}$$

  Hint: $\phi = \frac{\pi}{4}$ is obtained by figuring out the corresponding angle ϕ in which the upper hemisphere $x^2 + y^2 + z^2 = 2$ intersects the paraboloid $z = \sqrt{x^2 + y^2}$.

- Exercise 15. (i) If $x \in A \cup B$, then $x \in A$ or $x \in B$. If $x \in A$, then since A is open, there exists $r > 0$ such that $B_{r_A}(x) \subset A \subset A \cup B$. A similar case holds for $x \in B$.
  If $x \in A \cap B$, then $x \in A$ and $x \in B$. By openness of A and B, there exists $r_A > 0$ and $r_B > 0$ such that $B_{r_A}(x) \subset A$ and $B_{r_B} \subset B$. Thus, if we pick $r = \min\{r_A, r_B\}$, then we must have $B_r(x) \subset A \cap B$.
  (ii) If A and B are closed, then A^c and B^c are open. Notice that $A \cup B = (A^c \cap B^c)^c$ and $A \cap B = (A^c \cup B^c)^c$. Then, use (i) to arrive at the conclusion.
  (iii) $\partial A = \overline{A} \cap \overline{\mathbb{R}^n \setminus A}$.
  (iv) $\mathrm{int}(A) = \mathbb{R}^n \setminus \overline{A}$.

- Exercise 16. A "reverse" direction will greatly simply the computation. The corresponding $(u,v,w) = \Phi(x,y,z) = (x/a, y/b, z/c)$ and one can therefore show that $|\det(D\Phi)| = \frac{1}{abc}$. If we denote R to be the required region, then

$$\int\int\int_R 1\mathrm{d}V = \int\int\int_{B_1(0)} \frac{1}{1/abc}\mathrm{d}V = \int\int\int_{B_1(0)} abc\mathrm{d}V = \frac{4}{3}\pi abc. \tag{132}$$

  In the last equality, we have used the fact that the choice of substitution scales the ellipsoid down to a sphere of radius 1, and we know that the volume of such a sphere is $\frac{4\pi}{3}$.

- Exercise 17. A linear transformation T implies that there is a standard matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}. \tag{133}$$

  One can show by considering component wise, that the corresponding Jacobian matrix is given by

$$[J_T(\mathbf{x})] = A \tag{134}$$

  and thus

$$|\det(DT)| = |\det(J_T)| = |\det(A)| = |\det(T)| \tag{135}$$

  where the last equality is basically a formality, since the determinant of a linear transformation is defined to be the determinant of the corresponding standard matrix.
  Consequently, since $\det(T) = A$ which does not depend on the point of interest $\mathbf{x}$, we have the required conclusion for (ii).
  An example of the "component wise" argument: $(J_T(\mathbf{x}))_{11} = \frac{\partial}{\partial x_1}\sum_{i=1}^n A_{1i}x_i = A_{11}$.

5 Discussion 5

Summary for Lectures 11 - 12.

Definition 98. A **vector-valued** function is a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$, defined by

$$\mathbf{r}(t) = (x_1(t), \cdots, x_n(t))^T.\text{[a]} \tag{136}$$

[a]The superscript T here refers to transpose, so we are actaully looking at a column vector here as compared to a row vector.

Definition 99. The **graph** of a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is the set of points $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$ such that $f(\mathbf{x}) = \mathbf{y}$.

Thus, we can say the following:

Definition 100. A **curve** in $\mathbb{R}^n$ is the graph of a vector-valued function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^{n-1}$.

Intuitively, when we try to "graph" something, say from $\mathbb{R}^m$ to $\mathbb{R}^n$, we set aside "m" degree of freedoms/variables to "parameterize" that thing, such that if it lies in $\mathbb{R}^n$, each component can be parameterized by the same set of m variables. For example, since a curve is the graph of $\mathbf{r} : \mathbb{R} \to \mathbb{R}^{n-1}$, we have 1 degree of freedom to work with, parametrizing the remaining $n-1$ components with this common variable, and thus we can describe a curve in $\mathbb{R}^n$ by

$$(t, \mathbf{r}(t))^T = (t, x_2(t), x_3(t), ..., x_n(t))^T. \tag{137}$$

Note that this lies in $\mathbb{R}^n$, the number of variables used to parameterize the curve and the number of components to parameterize by, will have to sum up to n. This motivates the following:

Definition 101. A **surface** in $\mathbb{R}^n$ (for $n > 2$) is the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}^{n-2}$.

Intuitively, this means that we can "parameterize" each of the $n-2$ components with just 2 variables; ie

$$(x, y, f(x, y))^T = (x, y, x_3(x, y), ..., x_n(x, y))^T. \tag{138}$$

Generalizing to general number of variables/degree of freedom, we have the following:

Definition 102. A subset $M \subset \mathbb{R}^n$ is a **differentiable** $k-$**dimensional manifold embedded in** $\mathbb{R}^n$ if for all $\mathbf{x} \in M$, there exists an open neighborhood U such that $M \cap U$ is the graph of a C^1 mapping $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$.[a]

[a]Rigorously speaking, "graph of a C^1-mapping" should only hold for all points on $M \cap U$.

Examples:

- A differentiable **curve** in $\mathbb{R}^3$ is a $1-$dimensional manifold embedded in $\mathbb{R}^3$. This means that locally[3], it is the graph of a C^1 mapping $f : \mathbb{R}^1 \to \mathbb{R}^{3-1} = \mathbb{R}^2$.
- A differentiable **surface** in $\mathbb{R}^3$ is a $2-$dimensional manifold embedded in $\mathbb{R}^3$. This means that locally, it is the graph of a C^1 mapping $f : \mathbb{R}^2 \to \mathbb{R}^{3-2} = \mathbb{R}$.
- Disjoint unions of manifolds is a manifold.
- The figure-eight nor the union of two intersecting lines are differentiable curves in $\mathbb{R}^2$.

[3]Mathematically speaking, locally here means that for every given point $\mathbf{x} \in M$, we just have to find an open neighborhood U small enough such that in this neighborhood, the corresponding condition happens.

Definition 103. Let $X \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function. The **vanishing locus** of F (or just locus) is the set of points $V(F)$ where F vanishes. Mathematically,

$$V(F) = \{\mathbf{x} \in X : F(\mathbf{x}) = \mathbf{0}\}. \tag{139}$$

Definition 104. A map $f : X \to Y$ is **surjective** if for every $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.

Proposition 105. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a C^1-map. Fix a $\mathbf{z} \in \mathbb{R}^n$. Then, the following conditions on the derivative of F at $\mathbf{z}$, $[DF(\mathbf{z})]$, are equivalent:

- $[DF(\mathbf{z})]$ is surjective.
- At least one of the partial derivatives $\frac{\partial F}{\partial x_i}$ is non-zero.
- $[DF(\mathbf{z})] \neq [0, ..., 0]$.

Theorem 106. (Locally showing a vanishing locus is a differentiable manifold). Let $M \subset \mathbb{R}^n$ be a subset. Let $U \subset \mathbb{R}^n$ be open, and let $F : U \to \mathbb{R}^{n-k}$ be a C^1-mapping such that

$$M \cap U = \{\mathbf{z} \in U : F(\mathbf{z}) = \mathbf{0}\}. \tag{140}$$

If $[DF(\mathbf{z})]$ is a surjective map for every $z \in M \cap U$, then $M \cap U$ is a differentiable $k-$dimensional manifold embedded in $\mathbb{R}^n$.

Note: Surjectivity here refers to the linear map $[DF(\mathbf{z})]$ created with a given $\mathbf{z}$. Explicitly, suppose that we have some $\mathbf{z} \in M \cap U$, consider $[DF(\mathbf{z})] : \mathbb{R}^n \to \mathbb{R}^{n-k}$. Then, $[DF(\mathbf{z})]$ is surjective if for all $\mathbf{y} \in \mathbb{R}^{n-k}$, there exists $\mathbf{x} \in \mathbb{R}^n$ such that $[DF(\mathbf{z})](\mathbf{x}) = \mathbf{y}$. (Here, $[DF(\mathbf{z})](\mathbf{x})$ is understood as a matrix multiplication.)
Explicit Example: If for a given $\mathbf{z} \in \mathbb{R}^2$, we have $[DF(\mathbf{z})] = [2, 0] : \mathbb{R}^2 \to \mathbb{R}^1$. Then, let $y \in \mathbb{R}^1$ be given. Pick $\mathbf{x} = \begin{bmatrix} y/2 \\ 0 \end{bmatrix}$. Then, $[DF(\mathbf{z})]\mathbf{x} = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} y/2 \\ 0 \end{bmatrix} = y$. Thus, $[DF(\mathbf{z})] = [2, 0]$ is surjective.

Theorem 107. (Showing a vanishing locus is a differentiable manifold). Let $M \subset \mathbb{R}^n$ be a subset. If for every $\mathbf{z} \in M$, there exists open set U containing $\mathbf{z}$, and a C^1-mapping $F : U \to \mathbb{R}^{n-k}$ such that

$$M \cap U = \{\mathbf{z} \in U : F(\mathbf{z}) = \mathbf{0}\}, \tag{141}$$

then M is a differentiable $k-$manifold embedded in $\mathbb{R}^n$.

Theorem 108. (A differentiable manifold is locally a vanishing locus.) Conversely, let $M \subset \mathbb{R}^n$ be a differentiable $k-$dimensional manifold. Then, every point $\mathbf{z} \in M$ has a neighborhood $U \subset \mathbb{R}^n$ such that there exists a C^1-mapping $F : U \to \mathbb{R}^{n-k}$ with $[DF(\mathbf{z})]$ surjective and

$$M \cap U = \{\mathbf{z} \in U : F(\mathbf{z}) = \mathbf{0}\}. \tag{142}$$

Some remarks include:

- To show that a locus of $F = V(F)$ is a manifold, one can compute $[DF(\mathbf{z})]$ for any given $\mathbf{z} \in M \cap U$, and prove that $[DF(\mathbf{z})]$ is surjective. The conclusion follows from Theorem 106.
- With regards to the previous point, the surjectivity of $[DF(\mathbf{z})]$ is often easily shown using Proposition 105 or by definition of surjectivity.
- If $[DF(\mathbf{z})]$ is not surjective, one would have to prove this "directly", via Theorem 107. Thus, it is possible that $[DF(\mathbf{z})]$ is not surjective yet $V(F)$ is a differentiable manifold!

We will look at a simple explicit example below:

Example 109. Consider $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sin(x+yz)$, with $F:\mathbb{R}^3 \to \mathbb{R}$.

- Consider the set

$$M = \left\{ \begin{pmatrix} x \\ y \\ z \\ F(x,y,z) \end{pmatrix} \in \mathbb{R}^4 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \right\}. \tag{143}$$

  We can check that M is a $3-$dimensional manifold embedded in $\mathbb{R}^4$ as follows. For every given $(x,y,z,w) \in \mathbb{R}^4$, pick $U = \mathbb{R}^4$ (which is open, see 20), then $M \cap U = M$ is the graph of the mapping $F:\mathbb{R}^3 \to \mathbb{R}$ (this follows by the definition of a graph). For it to be a $3-$dimensional manifold embedded in $\mathbb{R}^4$, it suffices to check that F is C^1. Thus, one can compute:

$$\begin{aligned} \left[DF\begin{pmatrix} x \\ y \\ z \end{pmatrix} \right] &= \left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right] = [\cos(x+yz), z\cos(x+yz), y\cos(x+yz)] \\ &= \cos(x+yz)[1,z,y] \end{aligned} \tag{144}$$

  which we can see that each of the components here are continuous. Thus, all the partial derivatives of F are continuous and therefore, F is a C^1 mapping (see Discussion Supplement 4 for a recap on how to check that a mapping is C^1).

- Next, we consider the vanishing locus of F, $V(F)$, given by

$$V(F) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}. \tag{145}$$

  We claim that $V(F)$ is a $2-$dimensional surface embedded in $\mathbb{R}^3$. Appealing to Theorem 106, we shall pick $U \supset V(F)$ (with U open) and show that $\left[DF\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right]$ is surjective for every $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in M$. Indeed, from (144),

$$\left[DF\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right] = \cos(a+bc)[1,b,c]. \tag{146}$$

  Since $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in M$, by definition, we must have $\sin(a+bc) = 0$ and thus $\cos(a+bc) \neq 0$. This implies that the first component of $\left[DF\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right]$ is never zero, and thus by proposition 105, $\left[DF\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right]$ is surjective for arbitrary $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in M$. This implies that $V(F)$ is a $2-$dimensional surface embedded in $\mathbb{R}^3$. [a]

[a] Instead of using our intuition, we can also deduce the fact that it is a $2-$dimensional surface from the domain and codomain of $F:\mathbb{R}^1 = \mathbb{R}^{3-2}$ and thus $k = 2$ by looking at the specifics in Theorem 106.

Last but not least, we end off with some definitions for parameterization below:

Definition 110. A **strict** parameterization of a differentiable k-manifold $M \subset \mathbb{R}^n$ is a C^1-mapping $\gamma : U \subset \mathbb{R}^k \to M$ satisfying the following conditions:

1. U is an open set,
2. γ is injective [a], and surjects onto M, and
3. $[D\gamma(\mathbf{u})]$ is injective for all $\mathbf{u} \in U$.

[a] so that the manifold does not intersect with itself

Example: A strong parameterization of a curve in $\mathbb{R}^n$ is a C^1-mapping $\gamma : U \subset \mathbb{R} \to \mathbb{R}^n$ if it satisfies the following conditions:

1. U is an open interval,
2. γ is injective.
3. $[D\boldsymbol{\gamma}(u)] = \boldsymbol{\gamma}'(u)$ is injective for all $u \in U$.[4]

Example: If the manifold M is the graph of a single function $f(\mathbf{x}) = \mathbf{y}$, then M is parameterized by $\mathbf{x} \to \big(\mathbf{x}, f(\mathbf{x})\big)$
We shall end off with the following useful theorem for verifying conditions of a **strict** parameterization below:

Theorem 111. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A has the following equivalent properties

1. T is injective.
2. The only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.
3. If the equation $A\mathbf{x} = \mathbf{b}$ has a solution, then it is unique.
4. The columns of A are linearly independent.

Theorem 112. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A has the following equivalent properties

1. T is surjective.
2. The image of A is $\mathbb{R}^m$.
3. The columns of A spans $\mathbb{R}^n$.
4. The rows of A are linearly independent.
5. For every $\mathbf{b} \in \mathbb{R}^m$, there exists a $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$.

[4] γ here is bolded.

Let us look at an explicit example of a "strong parameterization".

Example 113. Let C be the upper half unit circle in $\mathbb{R}^2$ (excluding points $(\pm 1, 0)$). Thus, the parameterization $\gamma : (0, \pi) \to C$ given by

$$\gamma(\theta) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}. \tag{147}$$

Here, $U = (0, \pi)$ is an open subset of $\mathbb{R}$, and γ is bijective (both injective and surjective; one can check this by definition of injective and surjective) on this open interval. Furthermore,

$$\gamma'(\theta) = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{148}$$

for any given $\theta \in (0, \pi)$ (since $\sin$ and $\cos$ can't be simultaneously 0 at the same point). Thus, $[D\gamma(\theta)] = \gamma'(\theta)$ is injective for all $\theta \in (0, \pi)$ (since the only solution to $\begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is necessarily zero by the above observation, for any given $\theta \in (0, \pi)$). This implies that γ is a "strong parameterization" of the upper half unit circle in $\mathbb{R}^2$.

Exercises:

Exercise 18. Prove that the set

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x + x^2 + y^2 = 2 \right\} \tag{149}$$

is a differentiable $1-$dimensional manifold embedded in $\mathbb{R}^2$ (a.k.a a differentiable curve in $\mathbb{R}^2$).

Exercise 19. Show that the mapping:

$$\mathbf{g} : \begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \sin(uv) + u \\ u + v \\ uv \end{pmatrix} \tag{150}$$

is a parameterization of a differentiable (smooth) surface in $\mathbb{R}^3$ by showing the following:

(i) The image of $\mathbf{g}$ is contained in the locus S of the equation:

$$z = (x - \sin(z))(\sin(z) - x + y). \tag{151}$$

(ii) S is a smooth surface.

(iii) $\mathbf{g}$ maps $\mathbb{R}^2$ surjectively to S.

(iv) $\mathbf{g}$ is injective, and $\left[D\mathbf{g} \begin{pmatrix} u \\ v \end{pmatrix} \right]$ is injective for every $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$.

Exercise 20. Show that $\emptyset \subset \mathbb{R}^n$ and $\mathbb{R}^n$ are both open and closed in $\mathbb{R}^n$.

Partial Solutions/Hints:

- Exercise 18. Appeal to Theorem 106 and Proposition 105. To show that the set of points such that $[DF(x,y)] \neq (0,0)$, show that this is attained at $(-1/2,0)$ and that $(-1/2,0)$ does not lie on this curve.

- Exericse 19.

  (i) is done by direct substitution.

  (ii) uses a similar strategy as Exercise 18.

  (iii). To check that to solve for $\begin{pmatrix} u \\ v \end{pmatrix}$ such that $\mathbf{g}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sin(uv)+u \\ u+v \\ uv \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, just pick $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x-\sin(z) \\ y-x+\sin(z) \end{pmatrix}$. Note that when you are checking that it works, you are always free to use the fact that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S$ and thus (151) holds!

  (iv) can be checked by definition for $\mathbf{g}$ and appealing to Theorem 111 for $[D\mathbf{g}]$.

- Exercise 20: One can use the definition of open (in Discussion Supplement 4) and closed (in Discussion Supplement 1) to prove this directly. For the empty set, one can verify the statements for open and closed sets vacuously. Alternatively, one can use the results in the second exercise in Discussion Supplement 4.

6 Discussion 6

Summary for Lectures 13 - 15.

We shall recap the definition of **strict** parameterization as follows:

Definition 114. A **strict** parameterization of a differentiable k-manifold $M \subset \mathbb{R}^n$ is a C^1-mapping $\gamma : U \subset \mathbb{R}^k \to M$ satisfying the following conditions:

1. U is an open set,
2. γ is injective [a], and surjects onto M, and
3. $[D\gamma(\mathbf{u})]$ is injective for all $\mathbf{u} \in U$.

[a] so that the manifold does not intersect with itself

Example: A strong parameterization of a curve in $\mathbb{R}^n$ is a C^1-mapping $\gamma : U \subset \mathbb{R} \to \mathbb{R}^n$ if it satisfies the following conditions:

1. U is an open interval,
2. γ is injective.
3. $[D\boldsymbol{\gamma}(u)] = \boldsymbol{\gamma}'(u)$ is injective for all $u \in U$.[5]

Example: If the manifold M is the graph of a single function $f(\mathbf{x}) = \mathbf{y}$, then M is parameterized by $\mathbf{x} \to \big(\mathbf{x}, f(\mathbf{x})\big)$.

We shall end off with the following useful theorem for verifying conditions of a **strict** parameterization below:

Theorem 115. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A has the following equivalent properties

1. T is injective.
2. The only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.
3. If the equation $A\mathbf{x} = \mathbf{b}$ has a solution, then it is unique.
4. The columns of A are linearly independent.

Theorem 116. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A has the following equivalent properties

1. T is surjective.
2. The image of A is $\mathbb{R}^m$.
3. The columns of A spans $\mathbb{R}^n$.
4. The rows of A are linearly independent.
5. For every $\mathbf{b} \in \mathbb{R}^m$, there exists a $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$.

[5] γ here is bolded.

Let us look at an explicit example of a "strict parameterization".

Example 117. Let C be the upper half unit circle in $\mathbb{R}^2$ (excluding points $(\pm 1, 0)$). Thus, the parameterization $\gamma : (0, \pi) \to C$ given by

$$\gamma(\theta) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}. \tag{152}$$

Here, $U = (0, \pi)$ is an open subset of $\mathbb{R}$, and γ is bijective (both injective and surjective; one can check this by definition of injective and surjective) on this open interval. Furthermore,

$$\gamma'(\theta) = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{153}$$

for any given $\theta \in (0, \pi)$ (since $\sin$ and $\cos$ can't be simultaneously 0 at the same point). Thus, $[D\gamma(\theta)] = \gamma'(\theta)$ is injective for all $\theta \in (0, \pi)$ (since the only solution to $\begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is necessarily zero by the above observation, for any given $\theta \in (0, \pi)$). This implies that γ is a "strict parameterization" of the upper half unit circle in $\mathbb{R}^2$.

Now, let us consider a "softer" version of the **strict** parameterization above. To do so, we shall first introduce the following definitions:

Definition 118. Let $X \subset \mathbb{R}^n$ be a bounded subset. We say that X has $k-$**dimensional volume** 0 (ie $\text{vol}_k(X) = 0$) if

$$\lim_{N\to\infty} \sum_{C \in \mathcal{D}_N : C \cap X \neq \emptyset} \left(\frac{1}{2^N}\right)^k = 0. \tag{154}$$

Definition 119. An arbitrary subset $X \subset \mathbb{R}^n$ has $k-$dimensional volume 0 if for all R, the bounded set $X \cap B_R(\mathbf{0})$ has $k-$dimensional volume 0.

The following theorem can thus be verified:

Theorem 120. If $0 \leq m < k \leq n$ and $M \subset \mathbb{R}^n$ is a manifold of dimension m, then any closed subset $X \subset M$ has $k-$dimensional volume 0.

Intuitively, this means that the "higher dimensional volume" of a manifold (with that dimension) is 0. For instance, $\text{vol}_k(M) = 0$ if M is $m-$dimensional (with $m < k$).

The parameterization definition in full glory is as follows:

Definition 121. (General Parameterization.) Let $M \subset \mathbb{R}^n$ be a differentiable $k-$manifold. Let $A \subset \mathbb{R}^k$ be a subset such that $\text{vol}_k(\partial A) = 0$. Let $X \subset A$ be a subset such that $A \setminus X$ is open. Then, a C^1-mapping $\gamma : A \to \mathbb{R}^n$ parametrizes M if

(1) $M \subset \gamma(A)$,

(2) $\gamma(A \setminus X) \subset M$,

(3) $\gamma : A \setminus X \to M$ is injective,

(4) $[D\gamma(\mathbf{u}))]$ is injective for all $\mathbf{u} \in A \setminus X$, and

(5) X has $k-$dimensional volume 0, as does $\gamma(X) \cap C$ for any compact subset $C \subset M$.

If we pick $X = \partial A$, we then have the following particular case:

Definition 122. (Parameterization with possible overlap at the boundary.) Let $M \subset \mathbb{R}^n$ be a differentiable $k-$manifold. Let $D \subset \mathbb{R}^k$ be a connected subset such that $\text{vol}_k(\partial D) = 0$. (Here, we are choosing $A = D$ and $X = \partial D$. Recall that $A \setminus \partial A = A^\circ$ (or $\text{int}(A)$) is guaranteed to be open.) Then, a C^1-mapping parametrizes M if

(1) $M \subset \gamma(D)$,

(2) $\gamma(D^\circ) \subset M$,

(3) $\gamma : D^\circ \to M$ is injective,

(4) $[D\gamma(\mathbf{u}))]$ is injective for all $\mathbf{u} \in D^\circ$, and

(5) $\gamma(\partial D) \cap C$ for any compact subset $C \subset M$ has $k-$dimensional volume 0.
(It is guaranteed that $\text{vol}_k(X) = \text{vol}_k(\partial D) = 0$ by the hypothesis of the definition.)

To talk about connected sets, we shall introduce the following definitions (these are optional - probably just for your enrichment/for the mathematical enthusiasts)

Definition 123. Let $X \subset \mathbb{R}^n$. We say that Y is **open relative to** X if there exists an open set $Z \subset \mathbb{R}^n$ such that $Y = Z \cap X$.

Definition 124. We say that a subset $X \subset \mathbb{R}^n$ is **disconnected** if there exists **disjoint** sets $U, V \subset \mathbb{R}^n$, open relative to X, such that $X = U \cup Y$.

Definition 125. We say that a subset $X \subset \mathbb{R}^n$ is **connected** if it is not disconnected.

Example 126.

- $Y = [1, 2) \subset \mathbb{R}$ is open relative to $X = [1, 3]$. This is because there exists an open set in $\mathbb{R}$, given by $Z = (0, 2)$, such that $Y = [1, 2) = (0, 2) \cap [1, 3]$.
- Any subset $X \subset \mathbb{R}^n$ is always open relative to itself. This is because we can pick an open ball $Z \supset X$ so that $X = Z \cap X$.
- $X = [1, 2] \cup [3, 4]$ is disconnected. This is because $U = [1, 2]$ and $V = [3, 4]$ are disjoint and open relative to $X = [1, 2] \cup [3, 4]$.
(Since we can pick $U' = (0.9, 2.1)$ and $V' = (2.9, 4.1)$, both open, such that $U' \cap X = U$ and $V' \cap X = V$.)

We then end off with the following theorems:

Theorem 127. All manifolds can be parametrized.

Theorem 128. (Inverse image of a manifold.) Let $M \subset \mathbb{R}^m$ be a differentiable $k-$dimensional manifold embedded in $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be a C^1-mapping. Then, let $f^{-1}(M)$ be the inverse image (pre-image) of M, mainly

$$f^{-1}(M) = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \in M\}. \tag{155}$$

If $[DF(\mathbf{z})]$ is surjective for every $\mathbf{z} \in f^{-1}(M)$, then $f^{-1}(M)$ is a differentiable $(n + k - m)-$dimensional manifold embedded in $\mathbb{R}^n$.

Note that the above generalizes the idea that a vanishing locus is a manifold, since given an f as described above, the vanishing locus is mathematically described as

$$V(f) = f^{-1}(\{\mathbf{0}\}) \tag{156}$$

in which $\{\mathbf{0}\}$ is a $0-$dimensional manifold (thus $k = 0$), and thus $V(f) = f^{-1}(\mathbf{0})$ is a $(n - m)-$dimensional manifold.

With that, we shall take a look at an example below:

Example 129. (Spherical Coordinates.) Show that the map

$$S : \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} \to \begin{pmatrix} r\sin\phi\cos\theta \\ r\sin\phi\sin\theta \\ r\cos\phi \end{pmatrix} \tag{157}$$

with $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$, parametrizes space by the distance from the origin r, polar angle θ, and the angle from the north pole ϕ.

We shall check the conditions as in Definition 122.

- Here, $M = \mathbb{R}^3 \subset \mathbb{R}^3$. (You can view this as a "change of coordinates" rather than a "distinct parametrization".)
- $D = [0, \infty) \times [0, 2\pi] \times [0, \pi]$, with (draw this in $\mathbb{R}^3$ to visualize this as a cuboid in $\mathbb{R}^3$, with the last face at $r = \infty$ and thus not included)

$$\begin{aligned} \partial D = &([0.\infty) \times \{0\} \times [0, \pi]) \\ &\cup ([0.\infty) \times \{2\pi\} \times [0, \pi]) \\ &\cup ([0.\infty) \times [0, 2\pi] \times \{0\}) \\ &\cup ([0.\infty) \times [0, 2\pi] \times \{\pi\}) \\ &\cup (\{0\} \times [0, 2\pi] \times [0, \pi]) \end{aligned} \tag{158}$$

  with each piece having $\mathrm{vol}_3 = 0$. Thus, $\mathrm{vol}_3(\partial D) = 0$.
- (1) $M = \mathbb{R}^3 \subset S(D)$. Indeed, for every $(x, y, z) \in M$, we can find a value of $(r, \theta, \phi) \in D$ such that $S\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. This is because since (x, y, z) can be written in spherical coordinates as $x = r\sin\phi\cos\theta, y = r\sin\phi\sin\theta, z = r\cos\theta$, we shall pick the value of (r, θ, ϕ) obtained from spherical coordinates and place it in the argument for S.
- (2): $S(D^\circ) \subset M = \mathbb{R}^3$. Take an element $(x, y, z) \in S(D^\circ)$. This implies that there exists $(r, \theta, \phi) \in D^\circ$ such that $S\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Equivalently, by the definition of our map S, we get

$$\begin{pmatrix} r\sin\phi\cos\theta \\ r\sin\phi\sin\theta \\ r\cos\phi \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{159}$$

  It remains to show that such an $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in M$. This follows from the fact that any point in $\mathbb{R}^3$ can be expressed in spherical coordinates, and thus such a tuple of spherical coordinates must correspond to some $M = \mathbb{R}^3$. (In fact, since we are only allowing the argument of S to vary of D° rather than the entire D, we might be missing certain pieces of $\mathbb{R}^3$ but that is okay as long as it is in $\mathbb{R}^3$.) (See Discussion Supplement 4 on spherical coordinates.)

- (3): $S : D^\circ \to \mathbb{R}^3$ is injective. This can be done by assuming that we have $(r_1, \theta_1, \phi_1), (r_2, \theta_2, \phi_2) \in D^\circ$, and consider $S(r_1, \theta_1, \phi_1) = S(r_2, \theta_2, \phi_2)$ equivalently given by

$$\begin{cases} r_1 \sin\phi_1 \cos\theta_1 = r_2 \sin\phi_2 \cos\theta_2 \\ r_1 \sin\phi_1 \sin\theta_1 = r_2 \sin\phi_2 \sin\theta_2 \\ r_1 \cos\phi_1 = r_2 \cos\phi_2. \end{cases} \tag{160}$$

Take the squares of each equation and using the fact that $\sin^2(x) + \cos^2(x) = 1$ for any $x \in \mathbb{R}$, we have

$$r_1^2 = r_2^2. \tag{161}$$

Since $r_1, r_2 \geq 0$, we can take the positive root to obtain $r_1 = r_2$. This implies that the equation simplifies to

$$\begin{cases} \sin\phi_1 \cos\theta_1 = \sin\phi_2 \cos\theta_2 \\ \sin\phi_1 \sin\theta_1 = \sin\phi_2 \sin\theta_2 \\ \cos\phi_1 = \cos\phi_2. \end{cases} \tag{162}$$

From the third equation and the corresponding domain $\phi \in (0, \phi)$ (recall that we are looking at D° so the boundary points are excluded!), equality of cosines must imply the equality of its argument (since $\cos$ is injective on $(0, \phi)$). Thus, this implies the equality of $\sin(\phi_1) = \sin(\phi_2)$, which means that we can drop the $\sin\phi$ terms in the second equation. Now, since $\sin\theta_1 = \sin\theta_2$ and $\cos\theta_1 = \cos\theta_2$ for $\theta \in (0, 2\pi)$, we must obtain that $\theta_1 = \theta_2$ necessarily. This implies that $(r_1, \theta_1, \phi_1) = (r_2, \theta_2, \phi_2)$.

- (4): We can compute $DS(r_0, \theta_0, \phi_0)$ as follows:

$$[DS(r_0, \theta_0, \phi_0)] = \begin{pmatrix} \sin\phi_0 \cos\theta_0 & -r_0 \sin\phi_0 \sin\theta_0 & r_0 \cos\phi_0 \cos\theta_0 \\ \sin\phi_0 \sin\theta_0 & r_0 \sin\phi_0 \cos\theta_0 & r_0 \cos\phi_0 \sin\theta_0 \\ \cos\phi_0 & 0 & -r_0 \sin\phi_0 \end{pmatrix}. \tag{163}$$

As an added bonus, this also shows that S is a C^1-mapping from $D^\circ \to M$!
One can compute that $\det[DS(r_0, \theta_0, \phi_0)] = r_0^2 \sin\phi_0$. This implies that the solution to $[DS(r_0, \theta_0, \phi_0)] \begin{pmatrix} \dot{r} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is $\begin{pmatrix} \dot{r} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. By the theorem in Discussion Supplement 5, we deduce that this linear transformation is injective.

- (5) can be easily checked since $\gamma(\partial D)$ refers to either disks (fixed θ or ϕ, and let one angle and one radius parameter varies) or spherical shells (fixed r, and let both θ and ϕ varies) in $\mathbb{R}^3$, in which each of them are of $3-$dimensional volume of 0 (let alone, its intersection with compact subsets of $\mathbb{R}^3$).

Example 130. (Spherical Shell in Spherical Coordinates.) Show that the map

$$S : \begin{pmatrix} \theta \\ \phi \end{pmatrix} \to \begin{pmatrix} \sin\phi \cos\theta \\ \sin\phi \sin\theta \\ \cos\phi \end{pmatrix} \tag{164}$$

with $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$, parametrizes spherical surface/shell of radius 1 from the origin, with polar angle θ, and the angle from the north pole ϕ.
We shall check the conditions as in Definition 122.

- Here, $M \subset \mathbb{R}^3$, with the manifold M given by the spherical shell.
- $D = [0, 2\pi] \times [0, \pi]$, with

$$\begin{aligned} \partial D = &(\{0\} \times [0, \pi]) \cup (\{2\pi\} \times [0, \pi]) \\ &\cup ([0, 2\pi] \times \{0\}) \cup ([0, 2\pi] \times \{\pi\}) \end{aligned} \tag{165}$$

with each piece having $\mathrm{vol}_{\underline{2}} = 0$. Thus, $\mathrm{vol}_2(\partial D) = 0$. (Note that it is $\underline{2}$ here, since we are actually parametrizing a $2-$dimensional manifold!)

- (1) $M \subset S(D)$. Indeed, for every $(x, y, z) \in M$, we can find a value of $(\theta, \phi) \in D$ such that $S\begin{pmatrix}\theta \\ \phi\end{pmatrix} = \begin{pmatrix}x \\ y \\ z\end{pmatrix}$. This is because since (x, y, z) on the spherical shell (with $r = 1$) can be written in spherical coordinates as $x = \sin\phi\cos\theta, y = \sin\phi\sin\theta, z = \cos\theta$, we shall pick the value of (θ, ϕ) obtained from spherical coordinates and place it in the argument for S.

- (2) $S(D^\circ) \subset M$. In fact, one can show that $S(D) = M$ (from (1)). This implies that $S(D^\circ) \subset M$ (by taking a subset D° of the domain D, we should get an output $S(D^\circ) \subset S(D) = M$)

- (3) $S : D^\circ \to M$ is injective. By definition of injectivity, suppose that we have $(\theta_1, \phi_1), (\theta_2, \phi_2) \in D^\circ$, then $S(\theta_1, \phi_1) = S(\theta_2, \phi_2)$ implies that we have

$$\begin{cases} \sin\phi_1\cos\theta_1 = \sin\phi_2\cos\theta_2 \\ \sin\phi_1\sin\theta_1 = \sin\phi_2\sin\theta_2 \\ \cos\phi_1 = \cos\phi_2 \end{cases} \tag{166}$$

From the third equation and the corresponding domain $\phi \in (0, \phi)$ (recall that we are looking at D° so the boundary points are excluded!), equality of cosines must imply the equality of its argument (since $\cos$ is injective on $(0, \phi)$). Thus, this implies the equality of $\sin(\phi_1) = \sin(\phi_2)$, which means that we can drop the $\sin\phi$ terms in the second equation. Now, since $\sin\theta_1 = \sin\theta_2$ and $\cos\theta_1 = \cos\theta_2$ for $\theta \in (0, 2\pi)$, we must obtain that $\theta_1 = \theta_2$ necessarily. This thus implies that $(\theta_1, \phi_1) = (\theta_2, \phi_2)$, and thus the injectivity of S.

- (4): We can compute $DS(r_0, \theta_0, \phi_0)$ as follows:

$$[DS(\theta_0, \phi_0)] = \begin{pmatrix} -\sin\phi_0\sin\theta_0 & \cos\phi_0\cos\theta_0 \\ \sin\phi_0\cos\theta_0 & \cos\phi_0\sin\theta_0 \\ 0 & -\sin\phi_0 \end{pmatrix}. \tag{167}$$

One can show that the columns are linearly independent. Consider $\alpha, \beta \in \mathbb{R}$ and the vector equation:

$$\alpha\begin{pmatrix} -\sin\phi_0\sin\theta_0 \\ \sin\phi_0\cos\theta_0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} \cos\phi_0\cos\theta_0 \\ \cos\phi_0\sin\theta_0 \\ -\sin\phi_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{168}$$

As an added bonus, this also shows that S is a C^1-mapping from $D^\circ \to M$!
We say that the columns are linearly independent if $\alpha = \beta = 0$ necessarily. By comparing the third component, either we have $\sin\phi_0 = 0$ or $\beta = 0$. Since $\phi_0 \in (0, \pi)$ (reminder: we are working in D°, so the boundary points are excluded), the former is unattainable, and thus $\beta = 0$ necessarily. Looking at the first two components, we must have either $\alpha = 0$ or both $\sin\theta_0 = \cos\theta_0 = 0$. The latter is unattainable for any possible $\theta \in \mathbb{R}$ that you can think of, and therefore $\alpha = 0$ necessarily. By the theorem in Discussion Supplement 5, we deduce that this linear transformation is injective.

- (5) can be easily checked since $\gamma(\partial D)$ refers to either horizontal circles (for a fixed ϕ and let θ varies) or vertical half-circles (for a fixed θ and let ϕ varies) in $\mathbb{R}^2$, in which each of them are of $2-$dimensional volume of 0 (let alone, its intersection with compact subsets of $\mathbb{R}^2$).

Last but not least, we introduce the concept of tangent spaces below:

Definition 131. Let $M \subset \mathbb{R}^n$ be a $k-$dimensional manifold. For every $\mathbf{z}_0 \in M$, pick an open neighborhood U of $\mathbf{z}_0$ such that there exists a C^1-mapping $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$ (assuming $k < n$) that

$$M \cap U = \{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in U\} = \text{Graph}(f). \tag{169}$$

The **tangent space** to M at $\mathbf{z}_0$ is thus the graph of $[Df(\mathbf{z}_0)]$, given by

$$T_{\mathbf{z}_0}M = \{(\mathbf{x}, [Df(\mathbf{z}_0)](\mathbf{x})) : \mathbf{x} \in \mathbb{R}^k\} = \text{Graph}([Df(\mathbf{z}_0)]). \tag{170}$$

Note that for $T_{\mathbf{z}_0}M$, we are talking about the graph of a **linear transformation** $[Df(\mathbf{z}_0)]$. This is subtly different from that in the case of M, in which we were talking about the graph of a C^1-mapping f.

We can naturally generalize this for manifolds representing vanishing locus and represented by parametrization, as follows:

Theorem 132. (Tangent spaces for vanishing locus.) Let $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$ be a C^1-mapping in which a manifold M is given by $M = V(f)$. Suppose that $[DF(\mathbf{z}_0)]$ is surjective for some $\mathbf{z}_0 \in M$. Then, [a]

$$T_{\mathbf{z}_0}(M) = \ker[DF(\mathbf{z}_0)] = \{\mathbf{x} \in \mathbb{R}^n : [DF(\mathbf{z}_0)](\mathbf{x}) = \mathbf{0}\} \tag{171}$$

[a] ker here refers to the **kernel** of the mapping, which refers to a subset of the domain of the mapping that maps to the zero vector in the co-domain.

Theorem 133. (Tangent spaces of manifold represented by parametrization.) Let $U \subset \mathbb{R}^k$ be an open set, and let $\gamma : U \to \mathbb{R}^n$ be a parameterization of a manifold M. Then, for all $\mathbf{u} \in U$, [a]

$$T_{\gamma(\mathbf{u})}M = \text{Im}([D\gamma(\mathbf{u})]) = \{[D\gamma(\mathbf{u})](\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}. \tag{172}$$

[a] Im here refers to the **image**. Thus, we are looking at the image of the linear transformation $[D\gamma(\mathbf{u})] : \mathbb{R}^n \to \mathbb{R}^{n-k}$.

With that, we shall take a look some examples below:

Example 134. (i) Recall from one of the examples in Discussion Supplement 5 that the vanishing locus for

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sin(x+yz) \tag{173}$$

with $F : \mathbb{R}^3 \to \mathbb{R}$, is a smooth surface (2−dimensional manifold embedded in $\mathbb{R}^3$). We shall denote the surface as $M = V(F)$. Furthermore, we have computed the corresponding Jacobian matrix, which for a given $\mathbf{z}_0 = (a, b, c) \in M$, is given by

$$\left[DF\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right] = [\cos(a+bc), c\cos(a+bc), b\cos(a+bc)]. \tag{174}$$

To compute $T_{\mathbf{z}_0}M$, we have to compute $\ker[DF(\mathbf{z}_0)]$, that is, we want to describe the solution space of

$$[DF(\mathbf{z}_0)]\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{175}$$

for solutions $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$. [a] Indeed, from (175) and Theorem 132, we have

$$T_{\mathbf{z}_0}M = \left\{\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \in \mathbb{R}^3 : \cos(a+bc)\dot{x} + c\cos(a+bc)\dot{y} + b\cos(a+bc)\dot{z} = 0\right\}. \tag{176}$$

(ii) Furthermore, in the same Discussion Supplement, we have considered the manifold M' given by

$$M' = \left\{\begin{pmatrix} x \\ y \\ z \\ F(x,y,z) \end{pmatrix} \in \mathbb{R}^4 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3\right\}. \tag{177}$$

The corresponding parametrization is given by

$$\gamma\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ f(x,y,z) \end{pmatrix}. \tag{178}$$

One can check that this is indeed a parametrization (in view of Definition 122). Then, fix some $\mathbf{z}_0 = (a, b, c, d) \in M$. We can compute the following

$$[D\gamma(\mathbf{z}_0)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \cos(a+bc) & b\cos(a+bc) & c\cos(a+bc) \end{pmatrix}. \tag{179}$$

Then, in view of Theorem 133, we compute

$$[D\gamma(\mathbf{z}_0)]\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \cos(a+bc)\dot{x} + c\cos(a+bc)\dot{y} + b\cos(a+bc)\dot{z} \end{pmatrix}. \tag{180}$$

In linear algebra notation, as we allow $\dot{x}, \dot{y}$, and $\dot{z}$ to take any values on $\mathbb{R}$, we have

$$\begin{aligned}\operatorname{Im}([D\gamma(\mathbf{z}_0)]) &= \text{span}\left\{\begin{pmatrix}1\\0\\0\\\cos(a+bc)\end{pmatrix}, \begin{pmatrix}0\\1\\0\\c\cos(a+bc)\end{pmatrix}, \begin{pmatrix}0\\0\\1\\b\cos(a+bc)\end{pmatrix}\right\} \\ &= \left\{\dot{x}\begin{pmatrix}1\\0\\0\\\cos(a+bc)\end{pmatrix} + \dot{y}\begin{pmatrix}0\\1\\0\\c\cos(a+bc)\end{pmatrix} + \dot{z}\begin{pmatrix}0\\0\\1\\b\cos(a+bc)\end{pmatrix} : \dot{x}, \dot{y}, \dot{z} \in \mathbb{R}\right\}.\end{aligned} \tag{181}$$

[a] It is a convention to use the standard euclidean alphabets with a dot on top of it to symbolize "small increments", consistent with notations from Physics.

Exercises:

Exercise 21. (Spherical disks in Spherical Coordinates.) Show that the map

$$S : \begin{pmatrix} r \\ \phi \end{pmatrix} \to \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} \tag{182}$$

with $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$, parametrizes $\mathbb{R}^2 \times \{0\}$ by the distance from the origin r and the polar angle θ (with $\phi = \frac{\pi}{2}$).

Exercise 22. Consider the vanishing locus for the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = x + x^2 + y^2 - 2. \tag{183}$$

Here, $V(f)$ describes a curve in $\mathbb{R}^2$.

(i) Verify that $V(f)$ is indeed a smooth curve in embedded in $\mathbb{R}^2$.

(ii) Determine the equation for the tangent line to this curve at $(1, 0)$.

(iii) Compute the tangent space $T_{(1,0)}V(f)$.

Exercise 23. (Bonus - Connected subsets in $\mathbb{R}^2$). Consider $X = \overline{B_1((0,0))}$ and $Y = B_1((1,1))$. Show that $X \cup Y$ is not a connected subset of $\mathbb{R}^2$.

Partial Solutions/Hints:

- Exercise 21. A close analog would be to follow Example 130.
- Exericse 22.
  - (i) Compute $[Df(x,y)] = [1+2x, 2y]$ and show that it is surjective for all $(x,y) \in V(f)$. Either do this directly, or consider $g(x,y) = x + x^2 + y^2$ and see that it only has a critical at $x = -\frac{1}{2}, y = 0$, that does not lie on $V(f)$ (by direct verification). The logic then follows from Challenge Problem 2 Questions 4 and 5.
  - (ii) In a standard calculus class, this can be done by expressing y in terms of x. Here, we have $y = \pm\sqrt{2 - x - x^2}$. However, note that $\frac{\mathrm{d}y}{\mathrm{d}x}$ blows up at $(1,0)$. Instead, we shall compute $\frac{\mathrm{d}x}{\mathrm{d}y}$. In fact, this can be done explicitly by taking $\frac{\mathrm{d}}{\mathrm{d}y}$ on $x + x^2 + y^2 - 2 = 0$. This yields
  $$\frac{\mathrm{d}x}{\mathrm{d}y} + 2x\frac{\mathrm{d}x}{\mathrm{d}y} + 2y = 0. \tag{184}$$
  Thus, at $(1,0)$, we solve to obtain $\frac{\mathrm{d}x}{\mathrm{d}y} = 0$. The corresponding tangent line is given by $x - 1 = \frac{\mathrm{d}x}{\mathrm{d}y}(1,0)(y-0)$, equivalently, $\boxed{x = 1}$.
  - (iii) By definition, $T_{(1,0)}V(f) = \ker([Df(1,0)]) = \ker([3,0])$. One can check that this is indeed given by span $\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \left\{\begin{pmatrix} 0 \\ \dot{y} \end{pmatrix} : \dot{y} \in \mathbb{R}\right\}$.
- Exercise 23. This is equivalent to showing that X and Y are relatively open in $X \cup Y$. This is true if you pick the open set $U = B_{1.01}((0,0))$ and $V = Y$ so that $X = U \cap X$ and U is open in $\mathbb{R}^2$, similarly, $Y = V \cap Y$ with V being open in $\mathbb{R}^2$. Furthermore, we just have to show that U and V are disjoint.

7 Discussion 7

Summary for Lectures 16 - 17.

In view of the fact that I was not able to cover an example on the computation of tangent spaces, you should refer back to the Discussion Supplement from the previous week and work through that example if you have the time!

The focus of this discussion is to compute line/surface intergrals and in general, integrals on manifolds.

Idea: At each point $\mathbf{x}$ on a manifold M, we can approximate the tangent plane by a small parallelpiped. Thus, to compute the volume of a manifold, one just have to sum up the volume of these parallelpiped infinitesimally over the manifold.

Thus, before we introduce the concept of "manifoldal" integration (ie line/surface integrals in particular), we shall introduce some key definitions below:

Theorem 135. (Volume of a parallelpiped in $\mathbb{R}^3$.) Let D be a parallelpiped spanned by $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$.[a] Then, the volume of D is given by[b]

$$\mathrm{vol}_3(D) = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \left|\det \left(\mathbf{u}|\mathbf{v}|\mathbf{w}\right)\right|. \tag{185}$$

[a]To be exact, if we mean span in the linear algebra sense, then we are getting the entire $\mathbb{R}^3$, in which the volume here would be $+\infty$. Spanned here refers to the parallelpiped formed by a linear combination of $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ such that the coefficients are at most 1. For example, the mathematical definition of D would be $D = \{\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} : 0 \leq \alpha + \beta + \gamma \leq 1\}$. Indeed, if we let $\mathbf{u} = (2, 0, 0), \mathbf{v} = (0, 2, 0)$, and $\mathbf{w} = (0, 0, 3)$, we get a cuboid in $\mathbb{R}^3$ with side lengths $2, 2$, and 3 and thus obtain a volume of $2 \times 2 \times 3 = 12$. If you compute the volume according to the definition above, you should also obtain 12.

[b]This can be understood as lining the column vectors up as columns side by side to form a matrix to ask the determinant for!

Definition 136. Given $T = [T_{ij}] \in M_{m\times n}(\mathbb{R})$, the **transpose** matrix $T^\top \in M_{n\times m}(\mathbb{R})$ is given by

$$T^\top = [M_{ji}]. \tag{186}$$

Theorem 137. Let D be a $k-$dimensional parallelpiped spanned by $\mathbf{v}_1, \cdots, \mathbf{v}_k$ in $\mathbb{R}^k$. Then, consider the $k \times k$ matrix given by

$$T := \left(\mathbf{v}_1|\dots|\mathbf{v}_k\right). \tag{187}$$

Then, we have

$$\mathrm{vol}_k(D) = |\det(T)| = \sqrt{\det(T^\top T)}. \tag{188}$$

Definition 138. Let D be a $k-$dimensional parallelpiped spanned by $\mathbf{v}_1, \cdots, \mathbf{v}_k$ in $\mathbb{R}^n$. Then, consider the $k \times n$ matrix given by

$$T := \left(\mathbf{v}_1|\dots|\mathbf{v}_k\right). \tag{189}$$

Then, we have

$$\mathrm{vol}_k(D) = \sqrt{\det(T^\top T)}. \tag{190}$$

Example 139. Consider the $2-$dimensional parallelpiped D in $\mathbb{R}^3$ spanned by

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \text{ and } \mathbf{v} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \tag{191}$$

we can compute

- The $3-$dimensional volume, $\mathrm{vol}_3(D) = 0$. (Since this is a $2-$dimensional surface and we are asking for the $3-$dimensional volume!)
- The $2-$dimensional volume, $\mathrm{vol}_2(D)$, can be computed as follows. Compute

  $$T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \tag{192}$$

  and thus

  $$T^\top = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}. \tag{193}$$

  Thus, we have

  $$T^\top T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 14 & 32 \\ 32 & 77 \end{pmatrix}. \tag{194}$$

  Therefore, we have

  $$\det(T^\top T) = \det \begin{pmatrix} 14 & 32 \\ 32 & 77 \end{pmatrix} = 14 \cdot 77 - 32^2 = 54. \tag{195}$$

  Thus, the $2-$dimensional volume is given by

  $$\mathrm{vol}_2(D) = \sqrt{\det(T^\top T)} = \sqrt{54}. \tag{196}$$

- On a similar note, if we compute

  $$TT^\top = \begin{pmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{pmatrix}, \tag{197}$$

  then $\det(TT^\top) = 0$. Indeed, we have $\det(T^\top T) = \det(TT^\top)$ so the order here is important!

Now, we proceed to define the volume of manifolds:

Definition 140. Let $M \subset \mathbb{R}^n$ be a differentiable $k-$dimensional manifold, let $A \subset \mathbb{R}^k$ be a set with well-defined volume, and let $\gamma : A \to \mathbb{R}^n$ be a parametrization of M.[a] Then, we have[b]

$$\begin{aligned}\mathrm{vol}_k(M) &:= \int_{\gamma(A)} \mathrm{d}V(\mathbf{x}) \\ &= \int_A \sqrt{\det([D\gamma(\mathbf{u})]^\top [D\gamma(\mathbf{u})])}\mathrm{d}V(\mathbf{u}).\end{aligned} \tag{198}$$

[a] It does not matter if A is compact, or just an interior of a compact set, or a mixture of both. As long as we pick X in the general definition of parametrization to be ∂A, we always have $\gamma(A)$ and A in the integral above. Note that this does not depends on say if we are integrating on $A \setminus X$ or A, since X has $k-$dimensional volume 0, and can be thought of "a set of measure 0" so this does not affect the integral.

[b] Here, I have written down the explicit dependence so that one thing is clear; in the first expression, we are looking at the $n-$dimensional volume $\mathrm{d}V(\mathbf{x})$ while in the second, we are looking at the $k-$dimensional volume $\mathrm{d}V(\mathbf{u})$. Here, $\gamma : \mathbf{u} \in A \to \mathbf{x} \in \mathbb{R}^n$.

In view of this, we have the following definitions:

Definition 141. Let C be a curve in $\mathbb{R}^n$ parametrized by a C^1-function $\mathbf{r} : [a, b] \subset \mathbb{R} \to \mathbb{R}^n$. (Note that this implies that $[Dr(\mathbf{u})] = \mathbf{r}'(\mathbf{u})$ is injective for all $\mathbf{u} \in [a, b]$.)
Then, the **arc length** s of C is given by

$$\int_C \mathrm{d}s = \int_a^b ||\mathbf{r}'(t)||\mathrm{d}t. \tag{199}$$

Note that if $\mathbf{r}(t) = (r_1(t), \cdots, r_n(t))$, then $\mathbf{r}'(t) = (r_1'(t), \cdots, r_n'(t))$ and $||\mathbf{r}'(t)|| = \sqrt{r_1^2(t) + r_2^2(t) + \cdots r_n^2(t)}$.

For surfaces, let us consider the case in which $n = 3$ (ie manifolds embedded in $\mathbb{R}^3$).

Theorem 142. Let D be a parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$. Then,

$$\mathrm{vol}_2(D) = \text{area}(D) = ||\mathbf{u} \times \mathbf{v}||. \tag{200}$$

Thus, we have

Theorem 143. Let S be a surface parametrized by a C^1 function $\boldsymbol{\gamma} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Then, the area[a] of the parallelogram near a point $P \in S$ is given by

$$\text{area}(D) = \left|\left| \frac{\partial \boldsymbol{\gamma}(u, v)}{\partial u} \times \frac{\partial \boldsymbol{\gamma}(u, v)}{\partial v} \right|\right|. \tag{201}$$

Thus, we have

$$\int_S \mathrm{d}A = \int_U \left|\left| \frac{\partial \boldsymbol{\gamma}(u, v)}{\partial u} \times \frac{\partial \boldsymbol{\gamma}(u, v)}{\partial v} \right|\right| \mathrm{d}A. \tag{202}$$

[a] Similar to Theorem 135, area here is understood in the span way, in the sense that for every unit increase in each component in u and v, this is the amount that "parallelogram" gets scaled by. As understood in the sense of "Jacobian", this is really clear since we can understand this as $\mathrm{d}x\mathrm{d}y\mathrm{d}z$ " = " $\left|\frac{\partial(x,z,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v$, where $\left|\frac{\partial(x,z,y)}{\partial(u,v)}\right|$ represents the "Jacobian" in this case.

For cases in which $n \neq 3$, we will have to fall back to Definition 140.

Before we end off with a myraid of examples, one should note that we can generalize this to integrals of functions on manifolds (in particular, curves and surfaces) as follows:

Theorem 144. (Scalar Line Integral.) Let C be a curve in $\mathbb{R}^n$ parametrized by a C^1 function $\mathbf{r} : [a, b] \subset \mathbb{R} \to \mathbb{R}^n$. If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then we write the line integral as $\int_C f \mathrm{d}s$ and it can be computed as follows:

$$\int_C f \mathrm{d}s = \int_a^b f(\mathbf{r}(t))||\mathbf{r}'(t)||\mathrm{d}t. \tag{203}$$

Theorem 145. (Surface Integral in $\mathbb{R}^3$.) Let S be a surface in $\mathbb{R}^n$ parametrized by a C^1 function $\boldsymbol{\gamma} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. If $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous, then we write the surface integral as $\int_S f \mathrm{d}A$ and it can be computed as follows:

$$\int\int_S f \mathrm{d}A = \int\int_D f(\gamma(u, v)) \left|\left| \frac{\partial \boldsymbol{\gamma}(u, v)}{\partial u} \times \frac{\partial \boldsymbol{\gamma}(u, v)}{\partial v} \right|\right| \mathrm{d}A(u, v). \tag{204}$$

Theorem 146. (Integrating functions over manifolds.) Let $M \subset \mathbb{R}^n$ be a differentiable $k-$dimensional manifold in $\mathbb{R}^n$, and $A \subset \mathbb{R}^k$ be a set with well-defined volume, and let $\gamma : A \to \mathbb{R}^n$ be a parametrization of M. Let $f : M \to \mathbb{R}$ be a function. We say that f is integrable over M if the integral on the right hand side of the following exists:

$$\int_M f \mathrm{d}V := \int_A f(\gamma(\mathbf{u})) \sqrt{\det([D\gamma(\mathbf{u})]^\top [D\gamma(\mathbf{u})])} \mathrm{d}V(\mathbf{u}) \tag{205}$$

Example 147. Arc length of a semicircle. Consider an upper half (open) circle in $\mathbb{R}^2$ centered at the origin with radius r as follows. The following parametrizes this semicircle:

$$\gamma(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \tag{206}$$

for $0 < t < \pi$. (One can check that this indeed parametrizes the upper half semicircle. As checking that a parametrization is indeed one takes a lot of effort, you should only check it in the exam if the question requires you to!)
We can compute the arc length by Definition 141 as follows.

- $\gamma'(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$.
- $||\gamma'(t)|| = \sqrt{\sin^2(t) + \cos^2(t)} = 1$.
- Thus, we have
$$\int_C \mathrm{d}S = \int_0^\pi ||\gamma'(t)||\mathrm{d}t = \int_0^\pi \mathrm{d}t = \pi. \tag{207}$$

Example 148. Surface Integral - Area **of** a graph of a function $f(x, y)$ in $\mathbb{R}^2$. Consider $f(x, y) = x^2 + y^3$ with $f : \mathbb{R}^2 \to \mathbb{R}$. What is the area of the graph of $f(x, y)$ above the square $(0, 1) \times (0, 1)$?

Suggested Solution: We can parametrize the surface by

$$\gamma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x^2 + y^3 \end{pmatrix}. \tag{208}$$

The required area is given by

$$\int\int_S \mathrm{d}A = \int_0^1 \int_0^1 \left| \frac{\partial\gamma(x, y)}{\partial x} \times \frac{\partial\gamma(x, y)}{\partial y} \right| \mathrm{d}x\mathrm{d}y. \tag{209}$$

Thus, we have

$$\begin{aligned} \left| \frac{\partial\gamma(x, y)}{\partial x} \times \frac{\partial\gamma(x, y)}{\partial y} \right| &= \left| \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 3y^2 \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} -2x \\ -3y^2 \\ 1 \end{pmatrix} \right| \\ &= \sqrt{1 + 4x^2 + 9y^2} \end{aligned} \tag{210}$$

and the required area is given by

$$\int_0^1 \int_0^1 \sqrt{1 + 4x^2 + 9y^2}\mathrm{d}x\mathrm{d}y. \tag{211}$$

Note that it is challenging to compute this integral analytically. Numerically, we have that the integral gives 1.9322. □

Example 149. (Hubbard 5.3.6, modified.) Let S be part of the paraboloid of revolution $z = x^2 + y^2$ where $z < 9$.

(i) Verify the numerical values of the following integrals:

$$\int_0^3 r^3\sqrt{4r^2+1}\mathrm{d}r = \frac{1}{120}\left(1 + 1961\sqrt{37}\right) \tag{212}$$

and

$$\int_0^3 r^5\sqrt{4r^2+1}\mathrm{d}r = \frac{1}{840}\left(-1 + 87949\sqrt{37}\right). \tag{213}$$

(ii) Find a parametrization for S (you do not have to verify rigorously that it is indeed a parametrization of S).

(iii) Compute the integral $\int\int_S (x^2 + y^2 + 3z^2)\mathrm{d}A$ using the general formula in Theorem 146.

Suggested Solutions:

(i). This can be computing using an appropriate substitution. Note that $\frac{\mathrm{d}}{\mathrm{d}r}(4r^2+1) = 8r$, so we first use the substitution $R = 4r^2 + 1$ (thus $\mathrm{d}R = 8r\mathrm{d}r$) to obtain

$$\begin{aligned}\int_1^{37} r^3\sqrt{R}\frac{\mathrm{d}R}{8r} &= \int_1^{37} r^2\sqrt{R}\frac{\mathrm{d}R}{8} \\ &= \frac{1}{8}\int_1^{37}\left(\frac{R-1}{4}\right)\sqrt{R}\mathrm{d}R \\ &= \frac{1}{32}\int_1^{37} R^{\frac{3}{2}} - R^{\frac{1}{2}}\mathrm{d}R \\ &= \frac{1}{32}\left(\frac{2R^{\frac{5}{2}}}{5} - \frac{2R^{\frac{3}{2}}}{3}\right)\Bigg|_{R=1}^{R=37} \\ &= \frac{1}{120}\left(1 + 1961\sqrt{37}\right).\end{aligned} \tag{214}$$

A similar computation can be used to verify the second integral.

(ii). The corresponding parametrization is given by $\gamma : [0,3]\times[0,2\pi] \to S \subset \mathbb{R}^3$ where

$$\gamma\begin{pmatrix} r \\ \theta\end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r^2\end{pmatrix}. \tag{215}$$

(iii). We can do the integration as follows:

- Compute $[D\gamma(r,\theta)] = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ 2r & 0 \end{pmatrix}$.
- Now, compute

$$\begin{aligned} [D\gamma(r,\theta)]^\top [D\gamma(r,\theta)] &= \begin{pmatrix} \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ 2r & 0 \end{pmatrix} \\ &= \begin{pmatrix} 4r^2+1 & 0 \\ 0 & r^2 \end{pmatrix}. \end{aligned} \tag{216}$$

- $\sqrt{\det([D\gamma(r,\theta)]^\top [D\gamma(r,\theta)])} = r\sqrt{4r^2+1}$.
- Now, the integral becomes

$$\begin{aligned} \int\int_S (x^2+y^2+7z^2)\mathrm{d}A &= \int_0^3 \int_0^{2\pi} (r^2+7r^4) r\sqrt{4r^2+1}\mathrm{d}\theta\mathrm{d}r \\ &= 2\pi\left(\frac{1}{120}\left(1+1961\sqrt{37}\right) + 7\times\frac{1}{840}\left(-1+87949\sqrt{37}\right)\right) \\ &= \frac{2997\sqrt{2}\pi}{2}. \end{aligned} \tag{217}$$

Exercises:

Exercise 24. With respect to Example 148, consider instead that we have appealed to the general definition of the volume of a 2−dimensional manifold as in Definition 140. The two approaches will agree as long as the following is true:

$$\left|\frac{\partial\gamma(x,y)}{\partial x}\times\frac{\partial\gamma(x,y)}{\partial y}\right| = \sqrt{\det([D\gamma(x,y)]^\top[D\gamma(x,y)])}. \tag{218}$$

In the same example, we have also shown that

$$\left|\frac{\partial\gamma(x,y)}{\partial x}\times\frac{\partial\gamma(x,y)}{\partial y}\right| = \sqrt{1+4x^2+9y^2}. \tag{219}$$

Thus, it remains to show that

$$\sqrt{\det([D\gamma(x,y)]^\top[D\gamma(x,y)])} = \sqrt{1+4x^2+9y^2}. \tag{220}$$

By direct computation, show that this is indeed true for the parametrization γ as defined in Example 148.

Exercise 25. (Hubbard Example 5.3.7.) Let p, q be two integers, and consider the curve in $\mathbb{R}^4$ parametrized by

$$\gamma(t) = \begin{pmatrix}\cos(pt)\\ \sin(pt)\\ \cos(qt)\\ \sin(qt)\end{pmatrix}, \quad 0\le t\le 2\pi. \tag{221}$$

Compute the arc length of the curve.

Exercise 26. (Hubbard Exercise 5.3.18.) A gas has density $\rho(x,y,z) = \frac{C}{r}$ where $r = \sqrt{x^2+y^2+z^2}$ for some constant $C > 0$. If $0 < a < b$, what is the mass of the gas between the concentric spheres $r = a$ and $r = b$?

Note that the mass of a gas in a region D is given by

$$\int\int\int_D \rho(x,y,z)\mathrm{d}V(x,y,z). \tag{222}$$

Exercise 27. (Hubbard 5.3.8.) Compute the surface area of the part of the paraboloid of revolution $z = x^2+y^2$ where $z \le 1$.

Exercise 28. (Hubbard 5.3.15, modified.)
Consider the parametrization of the surface of a unit sphere in $\mathbb{R}^4$ by the following map:

$$\gamma\begin{pmatrix}\theta\\ \varphi\\ \psi\end{pmatrix} = \begin{pmatrix}\cos\psi\cos\varphi\cos\theta\\ \cos\psi\cos\varphi\sin\theta\\ \cos\psi\sin\varphi\\ \sin\psi\end{pmatrix}, \tag{223}$$

with φ,ψ,θ satisfying $-\pi/2\le\varphi\le\pi/2$,$-\pi/2\le\psi\le\pi/2$, and $0\le\theta<2\pi$. (One can check that this indeed parametrizes the surface of a unit sphere in $\mathbb{R}^4$.) Use this to compute the 3−dimensional volume of the surface of a unit sphere in $\mathbb{R}^4$.

<u>Partial Solutions/Hints:</u>

- Exercise 24. This is just a direct computational exercise. You can check that

$$[D\gamma(x,y)] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2x & 3y^2 \end{pmatrix}. \tag{224}$$

- Exercise 25. One can check that $||\gamma'(t)|| = \sqrt{p^2+q^2}$ independent of t. Thus, the arc length can be computed to give $2\pi\sqrt{p^2+q^2}$.

- Exercise 26. Use spherical coordinates in Discussion Supplement 4 (note that this is equivalent to a change of variables question since we are computing the integral of functions on a $3-$dimensional manifold embedded in $\mathbb{R}^3$, thinking of a region in $\mathbb{R}^3$ parametrized by spherical coordinates), and check that you did not miss out on the spherical Jacobian $r^2\sin(\phi)$. It should reduce to

$$\int_a^b \int_0^{2\pi} \int_0^{\pi} Cr\sin(\phi)\mathrm{d}\theta\mathrm{d}\phi = 2C\pi\left(b^2-a^2\right). \tag{225}$$

- Exercise 27. Similar to Example 149, but replace $x^2+y^2+7z^2$ with 1, and now $r \in [0,1]$. You should obtain $\frac{\pi}{6}\left(5\sqrt{5}-1\right)$. Note that you can repeat a similar argument (ie using an equivalent substitution) as to (i) to evaluate any integral that appears.

- Exercise 28. This is an exercise on computing $\sqrt{\det([D\gamma(\theta,\varphi,\psi)]^\top[D\gamma(\theta,\varphi,\psi)])}$ and the corresponding integral. One can check that $\sqrt{\det([D\gamma(\theta,\varphi,\psi)]^\top[D\gamma(\theta,\varphi,\psi)])} = \sin^2(\psi)\sin(\varphi)$ and the volume is thus $2\pi^2$.

8 Discussion 8

Summary for Lectures 18 - 19 and Concepts from Challenge Problem 3.

Definition 150. A **vector field** on $D \subset \mathbb{R}^n$ is a function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$. $\mathbf{F}$ assigns a vector $\mathbf{F}(p) \in \mathbb{R}^n$ to each point $p \in D$. Furthermore,

- $\mathbf{F}(p)$ is a **unit vector field** if $||\mathbf{F}(p)|| = 1$ for all $p \in D$.
- $\mathbf{F}(p)$ is a **radial vector field** if $\mathbf{F}(p) = \mathbf{F}(r)$ for all $p \in D$, where $r = ||p|| = \sqrt{x_1^2 + ... + x_n^2}$.[a]

[a] In $\mathbb{R}^3$, instead of being a function of x, y, and z, it only depends on $r = \sqrt{x^2+y^2+z^2}$. Example: $\mathbf{F}(x,y,z) = x$ is not radial, while $\mathbf{F}(x,y,z) = \begin{pmatrix} x^2+y^2+z^2 \\ 1 \end{pmatrix} = \begin{pmatrix} r^2 \\ 1 \end{pmatrix}$ is radial.

Definition 151. Given a curve C,

- A continuous choice of a tangent vector on C is called an **orientation**.
- A curve with a chosen orientation is called an **oriented curve**.
- Going along the choice of direction is known as the **positive direction**, while going against the choice of direction is known as the **negative direction**.

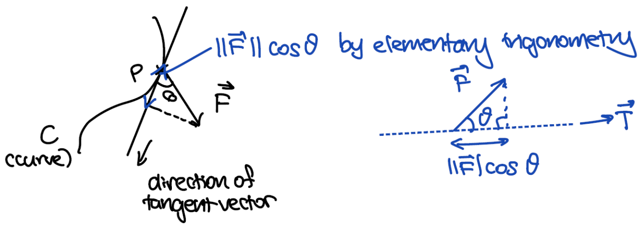
Definition 152. Given an oriented curve C, let $\mathbf{T}(p)$ be the unit tangent vector of C at the point p, pointing in the positive direction. Let $\mathbf{F}$ be a given vector field. Then, the **tangential component** of $\mathbf{F}$ at p is given by

$$\mathbf{T}(p) \cdot \mathbf{F}(p). \tag{226}$$

Remark 153. Note that this is consistent with our physical intuition of "tangential component". Since $\mathbf{T}$ is a unit tangent vector, then by the definition of dot product, we have

$$\mathbf{T} \cdot \mathbf{F} = ||\mathbf{T}||||\mathbf{F}|| \cos(\theta) = ||\mathbf{F}|| \cos(\theta), \tag{227}$$

where θ refers to the angle between $\mathbf{T}$ and $\mathbf{F}$.



Theorem 154. The line integral of $\mathbf{F}$ along an oriented curve C is the integral of the tangential component of $\mathbf{F}$:

$$\int_C \mathbf{F} \cdot \mathbf{r} := \int_C (\mathbf{F} \cdot \mathbf{T}) ds. \tag{228}$$

Remark 155. In physical terms, we say that $\int_C \mathbf{F} \cdot \mathbf{r}$ is the **work done by** a vector field on an object moving along the curve C, while $-\int_C \mathbf{F} \cdot \mathbf{r}$ is the **work done against** a vector field.

For explicit computations, we have

Theorem 156. Let $\mathbf{r}(t)$ be a positive oriented regular parametrization of an oriented curve C for $a \le t \le b$. Then,

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\mathrm{d}t. \tag{229}$$

Note that one can draw parallel between a vectorial line integral (in the theorem above), with a scalar line integral in the previous discussion if $\mathbf{F}$ assigns a scalar at each point (instead of a vector). To end off on line integrals, here are some properties of it:

Theorem 157. (Properties of Vector Line Integrals.) Let C be a C^1 oriented curve, and let $\mathbf{F}$ and $\mathbf{G}$ be vector fields.

- Linearity: $\int_C (\mathbf{F}+\mathbf{G}) \cdot \mathrm{d}r = \int_C \mathbf{F} \cdot \mathrm{d}r + \int_C \mathbf{G} \cdot \mathrm{d}r$, and $\int_C \lambda\mathbf{F} \cdot \mathrm{d}r = \lambda \int_C \mathbf{F} \cdot \mathrm{d}r$ for any $\lambda \in \mathbb{R}$.
- Reversing orientation: $\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.
- If C is the union of piecewise disjoint C^1 curves $C_1 \cup ... \cup C_m$, then we have

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + ... + \int_{C_m} \mathbf{F} \cdot \mathrm{d}\mathbf{r}. \tag{230}$$

Next, we shall introduce some of the common differential operators on scalar fields f and vector fields $\mathbf{F}$ as follows.

Definition 158.

- Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ where D is an open subset of $\mathbb{R}^n$. Then, the **gradient** of f, denoted by ∇f, is given by

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}. \tag{231}$$

- Let $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ where D is an open subset of $\mathbb{R}^n$. Then, the **divergence** of f, denoted by $\operatorname{div} \mathbf{F}$ or $\nabla \cdot \mathbf{F}$, is given by

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} \tag{232}$$

where

$$\mathbf{F} = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}. \tag{233}$$

- Let $\mathbf{F} : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ where D is an open subset of $\mathbb{R}^3$ (note here that $n = 3$). Then, the **curl** of f, denoted by $\operatorname{curl} \mathbf{F}$ or $\nabla \times \mathbf{F}$, is given by

$$\nabla \times \mathbf{F} = \begin{pmatrix} \frac{\partial}{\partial x_2}F_3 - \frac{\partial}{\partial x_3}F_2 \\ -\left(\frac{\partial}{\partial x_1}F_3 - \frac{\partial}{\partial x_3}F_1\right) \\ \frac{\partial}{\partial x_1}F_2 - \frac{\partial}{\partial x_2}F_1 \end{pmatrix} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \tag{234}$$

where (x_1, x_2, x_3) can be understood as (x, y, z) in $\mathbb{R}^3$.

Definition 159. A vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is conservative if there exists a differentiable function $f(x_1, \cdots, x_n)$ such that

$$\mathbf{F} = \nabla f \tag{235}$$

for some scalar (potential) function $f : \mathbb{R}^n \to \mathbb{R}$.

Theorem 160. (Fundamental Theorem for Conservative Vector Fields.) Let $\mathbf{F}$ be a conservative vector field on a domain D; that is, there exists a C^1 (smooth/continuously differentiable) f such that $\mathbf{F} = \nabla f$. Given any oriented curve C from P to Q, if $\mathbf{r}$ is a positively oriented parametrization along the curve C in D, then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(Q) - f(P). \tag{236}$$

In particular, $\mathbf{F}$ is path-independent.

Note the following result from Challenge Problem 3:

Theorem 161. For an arbitrary conservative vector field $\mathbf{F}$ on an open connected set D, any two potential functions $\mathbf{F}$ differ by a constant.[a]

[a]This generalizes the result from Challenge Problem 3, in which instead of demanding that D is an open disk, we could work with arbitrary open sets in $\mathbb{R}^n$ that are connected.

Thus, we have the following

Corollary 162. Let $\mathbf{F} = \nabla f$ be a conservative vector field in a domain D. If $\mathbf{r}$ is a positively oriented parametrization along a **closed** curve C in D, then we have

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0, \tag{237}$$

where the symbol $\oint_C$ refers to an integration along the closed loop C.

Lemma 163. The following is true for a smooth (or at least C^2; twice continuously differentiable) scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ or a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$:

- $$\nabla \times (\nabla f) = \mathbf{0}. \tag{238}$$

- $$\nabla \cdot (\nabla \times \mathbf{F}) = 0. \tag{239}$$

Theorem 164. (Curl of Conservative Vector Fields.) (Assume that $\mathbf{F}$ are sufficiently smooth.)

- In $\mathbb{R}^2$, if $\mathbf{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ is conservative, then $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ for all $(x, y) \in \mathbb{R}^2$.
- In $\mathbb{R}^2$, if $\mathbf{F}$ is conservative, then $\mathrm{curl}(F) = \nabla \times \mathbf{F} = 0$. Equivalently, $\mathbf{F}$ must satisfy the cross-partial relaton below for every $(x, y, z) \in \mathbb{R}^3$:[a]

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \text{ and } \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}. \tag{240}$$

[a]Or in the corresponding domain D.

The condition for $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ can be thought of as $\mathrm{curl}(\mathbf{G}) = \mathbf{0}$, where $\mathbf{G} = (F_1, F_2, 0) = (\mathbf{F}, 0)$.

Theorem 165. Let $\mathbf{F}$ be a vector field on a **simply-connected** domain D. If $\mathbf{F}$ satifies the cross-partial condition, then $\mathbf{F}$ is conservative.

Note that we can think of simply-connected domains as **connected** domains and are **without holes**. Furthermore, the requirement that D is simply-connected is necessary. Else, we can consider $\mathbf{F}(x,y) = \begin{pmatrix} -\frac{x}{x^2+y^2} \\ \frac{y}{x^2+y^2} \end{pmatrix}$ defined on $\mathbb{R}^2 \setminus \{0\}$. It satisfies the cross-partial relation but it is not conservative.
In the next few pages, we shall explore a myriad of examples on line integrals, conservative vector fields and its application on evaluating line integrals.

Example 166. For each of the oriented curves C in $\mathbb{R}^3$ below, compute the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ of the vector field

$$\mathbf{F}(x,y,z) = \begin{pmatrix} 3x^2 - 6yz \\ 2y + 3xz \\ 1 - 4xyz^2 \end{pmatrix} \tag{241}$$

with respect to (a suitable parametrization) $\mathbf{r}$.

(a) C is a straight line connecting $(0,0,0)$ to $(1,1,1)$.

(b) C is the curve parametrized by $\mathbf{r}(t) = (t, t^2, t^3)$ for $t \in [0,1]$.

For (a), you do not need to check that your choice of parametrization is indeed a parametrization.

(c) Deduce that $\mathbf{F}$ is not conservative.

Suggested Solutions:

(a) A suitable parametrization of the line segment would be $\mathbf{r}'(t) = (t,t,t)$ for $t \in [0,1]$. Using the formula in Theorem 156, we compute

- $\mathbf{r}'(t) = (1,1,1)$.
- $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (3t^2 - 6t^2, 2t + 3t^2, 1 - 4t^4) \cdot (1,1,1) = -3t^2 + 2t + 3t^2 + 1 - 4t^4 = 1 + 2t - 4t^4$. (Note that $x = y = z = t$ by our parametrization.)
- Then, $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\mathrm{d}t = \int_0^1 (1 + 2t - 4t^4)\mathrm{d}t = \frac{6}{5}$.

(b) Using the given parametrization, we compute

- $\mathbf{r}'(t) = (1, 2t, 3t^2)$.
- $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (3t^2 - 6t^5, 2t^2 + 3t^4, 1 - 4t^9) \cdot (1, 2t, 3t^2) = 6t^2 + 4t^3 - 12t^{11}$. (Note that $x = t, y = t^2,$ and $z = t^3$ by our parametrization.)
- Then, $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\mathrm{d}t = \int_0^1 (6t^2 + 4t^3 - 12t^{11})\mathrm{d}t = 2$.

(c) Note that for both parts, we are evaluating the vector line integral from $(0,0,0)$ to $(1,1,1)$ via two different curves ((a) - line segment, (b) - curve). As they both yield different values, this implies that the line integral is not path-independent, and thus $\mathbf{F}$ cannot be conservative.

Example 167. For each of the following vector fields $\mathbf{F}$ in $\mathbb{R}^3$, determine whether if it is a conservative vector field. If it is, determine **all** possible potential functions f for $\mathbf{F}$.

(a) $\mathbf{F}(x,y,z) = \begin{pmatrix} 2xy^3 \\ x^2z^3 \\ 3x^2yz^2 \end{pmatrix}$.

(b) $\mathbf{F}(x,y,z) = \begin{pmatrix} 3x^2y \\ x^3 \\ 5 \end{pmatrix}$.

Suggested Solutions:

Note that $\mathbb{R}^3$ is a simply connected region. Thus, Theorem 164 and Theorem 165 acts in both direction (ie $\mathbf{F}$ is conservative **if and only if it** satisfies the cross-partial relation.)

(a) Compute $\nabla \times \mathbf{F}$ to obtain

$$\nabla \times \mathbf{F} = \begin{pmatrix} 0 \\ -6xyz^2 \\ 2xz^3 - 6xy^2 \end{pmatrix} \neq \mathbf{0}. \tag{242}$$

Since it does not satisfy the cross-partial relation (ie $\nabla \times \mathbf{F} - \mathbf{0}$ for every $(x,y,z) \in D = \mathbb{R}^3$), then $\mathbf{F}$ is not conservative.

(b) One can check that $\nabla \times \mathbf{F} = \mathbf{0}$. It remains to find **all** possible potential functions f. Here, we propose two methods, in which the reader is free to pick up on whatever method that works.

Method 1: Solving a linear system of (first order) PDEs (Partial Differential Equations). If we would like $\mathbf{F} = \nabla f$ for some f, this implies that we must have the following system of (differential) equations below

$$\begin{aligned} \frac{\partial f}{\partial x} &= F_x = 3x^2y \\ \frac{\partial f}{\partial y} &= F_y = x^3 \\ \frac{\partial f}{\partial z} &= F_z = 5. \end{aligned} \tag{243}$$

- Integrating the first equation partially with respect to x, we have

$$f(x,y,z) = x^3y + g_1(y,z).$$

  Here, the arbitrary constant g_1 may depend on y and z (which is possible since upon an application of $\frac{\partial}{\partial x}$, it disappears).

- Similarly, we can integrate the second equation partially with respect to y to obtain

$$f(x,y,z) = x^3y + g_2(x,z).$$

  (since we did a partial integration with respect to y, then the arbitrary constant will have to depend on the other two variables x, z.)

- Analogously, we have

$$f(x,y,z) = 5z + g_3(x,y)$$

  from the third equation.

- By inspection, we can pick $g_1(y,z) = 5z$, $g_2(x,z) = 5z$, and $g_3(x,y) = x^3y$.

This implies that we have

$$f(x, y, z) = x^3y + 5z. \tag{244}$$

Since all potential functions differ by a constant on an open connected domain (in which $\mathbb{R}^3$ is), then all the possible potential functions are given by

$$f(x, y, z) = x^3y + 5z + C \tag{245}$$

for any arbitrary constant C. [a]

Method 2: Formulating a line integral to an arbitrary point (x, y, z). Since $\mathbf{F}$ is conservative, then we define

$$f(x, y, z) = \int_{C(x,y,z)} \mathbf{F} \cdot \mathrm{d}\mathbf{r}, \tag{246}$$

where $C(x, y, z)$ is the line segment from $(0, 0, 0)$ to (x, y, z), in which we can parametrize it by $\mathbf{r}(t) = (tx, ty, tz) = (x(t), y(t), z(t))$ for $t \in [0, 1]$. Note that such a definition is legal since the starting and ending points are fixed, and by the fact that $\mathbf{F}$ is conservative, it should not depend on the path taken and thus a line segment will do the job. Now, all we are left to do is to compute the line integral on the right.

- $\mathbf{r}'(t) = (x, y, z)$.
- Then, we have

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= (3(x(t))^2(y(t)), (x(t))^3, 5) \cdot (x, y, z) \\ &= (3(tx)^2(ty), (tx)^3, 5) \cdot (x, y, z) \\ &= 4x^3yt^3 + 5z.\end{aligned}$$

  (Note that $x(t) = tx, y(t) = ty,$ and $z(t) = tz$ by our parametrization.)
- Finally, we compute $\int_{C(x,y,z)} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\mathrm{d}t = \int_0^1 (4x^3yt^3 + 5z)\mathrm{d}t = x^3y + 5z$.

Note that the above is **a** potential function and not all possible potential functions. To conclude, we appeal to the fact that all potential functions differ by a constant, and thus, we have that potential functions are given by

$$f(x, y, z) = x^3y + 5z + C \tag{247}$$

for any arbitrary constant C.

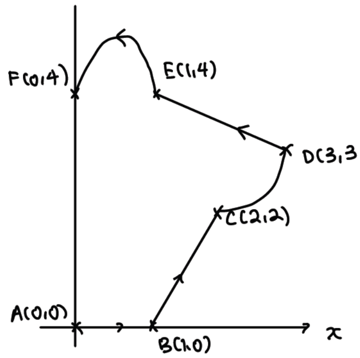
[a] Here, we note that we could have chosen $g_1 = 5z + C, g_2 = 5z + C$, and $g_3 = x^3y + C$ and arrive at this conclusion instantly.

Example 168. Consider the following path ABCDEF as shown in the diagram below. Note that from C to D, this is connected by the curve $y = (x-2)^2$ for $2 \le x \le 3$. From E to F, it is connected by the sin curve $y = \sin(\pi x) + 4$ for $0 \le x \le 1$. Now, consider the vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathbf{F}(x,y) = \begin{pmatrix} y \\ x+1 \end{pmatrix}. \tag{248}$$

Evaluate the line integral along the oriented curve ABCDEF

$$\int_{\text{ABCDEF}} \mathbf{F} \cdot d\mathbf{r}. \tag{249}$$



Suggested Solution:

Note that $\mathbf{F}$ is conservative, since $\frac{\partial F_1}{\partial y} = 1 = \frac{\partial F_2}{\partial x}$. We can obtain the potential function f such that $\mathbf{F} = \nabla f$ by solving the system of differential equations below:

$$\begin{aligned} \frac{\partial f}{\partial x} &= y \\ \frac{\partial f}{\partial y} &= x+1 \end{aligned} \tag{250}$$

Doing the partial integration in the corresponding variables, we get

$$\begin{aligned} f(x,y) &= xy + g_1(y) \\ f(x,y) &= xy + y + g_2(x). \end{aligned} \tag{251}$$

One can observe that a legal choice of potential function (pick $g_1 = y$ and $g_2 = 0$) would be

$$f(x,y) = xy + y. \tag{252}$$

Since $\mathbf{F}$ is conservative, by the Fundamental Theorem of Conservative Vector Fields, we have that

$$\int_{\text{ABCDEF}} \mathbf{F} \cdot d\mathbf{r} = f(F) - f(A). \tag{253}$$

Thus, we have

$$\int_{\text{ABCDEF}} \mathbf{F} \cdot d\mathbf{r} = f(0,4) - f(0,0) = 4 - 0 = 4. \tag{254}$$

Exercises:

Exercise 29. Continuing from Example 166 with the vector field given in (241), compute the line integral along the polygonal line joining $(0,0,0),(0,0,1),(0,1,1),(1,1,1)$ in that order.

Exercise 30. Consider the following vector field in $\mathbb{R}^3$, given by

$$\mathbf{F}(x,y,z)=\begin{pmatrix} x+z \\ -(y+z) \\ (x-y) \end{pmatrix}. \tag{255}$$

Determine if it is a conservative vector field. If it is, determine all possible potential functions f for $\mathbf{F}$.

Exercise 31. (ML Inequality.) Let C be a smooth curve in $\mathbb{R}^n$, given by a parametrization $\mathbf{r}: I \to \mathbb{R}^n$ where $I \subset \mathbb{R}$ is a compact interval. Let $U \subset \mathbb{R}^n$ be an open set with $C = \mathbf{r}(I) \subset U$. Let $F: U \to \mathbb{R}^n$ be a continuous vector field on U. Suppose that $M \geq 0$ is a constant such that for all $p \in U$, $||\mathbf{F}(p)|| \leq M$. Show that

$$\left| \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} \right| \leq ML, \tag{256}$$

where L is the length of the curve, which is given by $\int_C 1 \mathrm{d}s$.

Exercise 32. Let $\mathbf{F}: \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field given by

$$\mathbf{F}(x,y,z) = \left(y^2\cos(x) + z^3, 2y\sin(x) - 4, 3xz^2 + 2\right). \tag{257}$$

Evaluate the work done by the vector field along the positively oriented curve C, which is a straight line going from $(0,1,-1)$ to $(\pi/2,-1,2)$.

Partial Solutions/Hints:

- Exercise 29. Break up the polygonal line into 3 line segments, with $(0,0,0) \to (0,0,1)$, $(0,0,1) \to (0,1,1)$, and $(0,1,1) \to (1,1,1)$. These three line segments are parametrized by

$$\begin{aligned} \mathbf{r}_1(t) &:= (0,0,t), t \in [0,1] \\ \mathbf{r}_2(t) &:= (0,t,1), t \in [0,1] \\ \mathbf{r}_3(t) &:= (t,1,1), t \in [0,1]. \end{aligned} \tag{258}$$

Hence, we have (using one of the properties for line integrals as in Theorem 157, we have

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{259}$$

- Exercise 30. One can follow a similar procedure as Example 167, and obtain that $\mathbf{F}$ is conservative with a potential function $f(x,y,z) = \frac{1}{2}\left(x^2 + 2xy - y^2 - 2yz\right) + C$ for arbitrary constant C.

- Exercise 31. We have

$$\begin{aligned} \left|\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}\right| &= \left|\int_I \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\mathrm{d}t\right| \text{ (by definition of line integrals)} \\ &\leq \int_I |\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)|\, \mathrm{d}t \text{ (by triangle inequality for integrals in } \mathbb{R}^n) \\ &\leq \int_I ||\mathbf{F}(\mathbf{r}(t))||||\mathbf{r}'(t)||\mathrm{d}t \text{ (by Cauchy-Schwarz Inequality for dot product)} \\ &\leq \int_I M||\mathbf{r}'(t)||\mathrm{d}t \text{ (by the given assumption)} \\ &= M\int_I ||\mathbf{r}'(t)||\mathrm{d}t \text{ (property of integrals allowing me to pull out a scalar)} \\ &= M\int_C 1\mathrm{d}s \text{ (by the definition of arc length, see the previous Discussion Supplement)} \\ &\leq ML \text{ (by the given assumption).} \end{aligned} \tag{260}$$

- Exercise 32. Note that work done by the vector field along an oriented curve C is defined as the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

  Check that $\mathbf{F}$ is conservative by computing $\nabla \times \mathbf{F}$. By the Fundamental Theorem of Conservative Vector Fields, we have $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(\pi/2,-1,2) - f(0,1,-1)$. It remains to find the corresponding expression for f. One can appeal to the techniques mentioned in the previous example, but as someone who loves PDE, I will only do Method 1 for this example. (You can check that Method 2 will give you the same solution). This means that we have to solve the system of PDEs below

$$\begin{aligned} \frac{\partial f}{\partial x} &= y^2\cos(x) + z^3 \\ \frac{\partial f}{\partial y} &= 2y\sin(x) - 4 \\ \frac{\partial f}{\partial z} &= 3xz^2 + 2. \end{aligned} \tag{261}$$

Doing the partial integration in the corresponding variables, we get

$$\begin{aligned} f(x,y,z) &= y^2\sin(x) + xz^3 + g_1(y,z) \\ f(x,y,z) &= y^2\sin(x) - 4y + g_2(x,z) \\ f(x,y,z) &= xz^3 + 2z + g_3(x,y). \end{aligned} \tag{262}$$

By observation, we can see that $g_1 = -4y + 2z$, $g_2 = xz^3 + 2z$, and $g_3 = y^2\sin(x) - 4y$. (Just "add" whatever that is missing that you can find from the other terms!) Thus, a potential function is given by

$$f(x,y,z) = y^2\sin(x) + xz^3 - 4y + 2z. \tag{263}$$

Thus, the required work done is given by

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(\pi/2, -1, 2) - f(0, 1, -1) = 15 + 4\pi. \tag{264}$$

9 Discussion 9

Summary for Lectures 20 - 21.

Definition 169. Given an oriented curve in $\mathbb{R}^2$, we say that the **positive direction across** C is the direction that goes from left to right from the perspective along the positive orientation of C.

Definition 170. Let $\mathbf{n}(p)$ be a unit vector normal to C at the point p, pointing in the positive direction across C. Let $\mathbf{F}$ be a vector field. The **normal component of F** at p is given by the dot product

$$\mathbf{n}(p) \cdot \mathbf{F}(p).$$

Thus, we can define a **flux integral** in $\mathbb{R}^2$ across a curve C with respect to a vector field $\mathbf{F}$ as follows:

Definition 171. The **flux integral** of a vector field $\mathbf{F}$ along an oriented curve C in $\mathbb{R}^2$ is the integral of the normal component of $\mathbf{F}$:

$$\int_C (\mathbf{F} \cdot \mathbf{n}) \mathrm{d}s. \tag{265}$$

Theorem 172. Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$[a] be a positively oriented regular parametrization of an oriented curve C, with $a \leq t \leq b$. Observe that $\mathbf{N}(t) = \langle y'(t), -x'(t) \rangle$ is normal to C.[b] Then,

$$\int_C (\mathbf{F} \cdot \mathbf{n}) \mathrm{d}s = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{N}(t)\ \mathrm{d}t. \tag{266}$$

[a] Note that $\langle a, b \rangle$ really just means $\begin{pmatrix} a \\ b \end{pmatrix}$. Thus, we will abuse notation and occasionally switch around, or even write it as (a, b) as long as the context is clear.

[b] Normal here means that $\mathbf{N}(t) \cdot \mathbf{r}'(t) = 0$.

Remark 173. Note that $\mathbf{n}(t) = \frac{\mathbf{N}(t)}{||\mathbf{N}(t)||}$ refers to the unit vector normal to C, while $\mathbf{N}(t)$ here reflects its full length. (That is, it might not be of unit length!)

Example 174. (Parametrizing a parabola in $\mathbb{R}^2$.) Find a positively oriented parametrization of the parabola $y = x^2$ for $0 \leq x \leq 1$, oriented from $(0, 0)$ to $(1, 1)$.

Suggested Solution: Set $\mathbf{r}(t) = \langle t, t^2 \rangle$ for $0 \leq t \leq 1$. Indeed, you can see that this parametrizes the parabola. To verify that this is a positively oriented parametrization of the parabola (see Definition in Discussion Supplement 8), we note that the continuous choice of tangent vector on C is defined to be along the parabola, from $(0, 0)$ to $(1, 1)$. As our parametrization is such that as t increases, we move along the curve from $(0, 0)$ to $(1, 1)$, such a parametrization corresponds to a positively oriented one. We shall leave the verification that this is indeed a parametrization as an exercise to the reader.

Remark: To do this in a rigorous fashion, one would have to refer to the exact mathematical definition of an orientation in Challenge Problem Set 4. Once you are done with that, try to see if you can prove that this is indeed a positive orientation in a rigorous sense.

The idea here is that flux integrals in $\mathbb{R}^n$ can be computed along $n-1$ dimensional manifolds. This allow us to generalize the idea of a flux integral to $\mathbb{R}^3$ across a surface as follows.

Definition 175. Let S be a surface in $\mathbb{R}^3$. An orientation S is a continuous choice of unit normal vector $\mathbf{n}(p)$ at each point p on S. We can then define the following terms:

- The **positive orientation across** S is in the direction of the normal vector $\mathbf{n}$.
- The **negative orientation across** S is in the direction of $-\mathbf{n}$.
- We say that a surface with a choice of orientation is an **oriented surface**.

Definition 176. Let $\mathbf{F}$ be a vector field in $\mathbb{R}^3$, and let $p = \mathbf{G}(u_0, v_0)$ be a point on an oriented surface S with parametrization G. The **normal component** of $\mathbf{F}$ at p is the dot product

$$\mathbf{F}(p) \cdot \mathbf{n}(p).$$

Definition 177. The **vector surface integral** over S is defined as

$$\int\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} := \int\int_S (\mathbf{F} \cdot \mathbf{n})\mathrm{d}S. \tag{267}$$

This is also known as the **flux** of $\mathbf{F}$ across S.

Remark 178. Compare with a vector line integral, in which we define

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} := \int_C (\mathbf{F} \cdot \mathbf{T})\mathrm{d}s.$$

Definition 179. For a fluid with velocity vector field $\mathbf{v}$, the flow rate across S is given by

$$\int\int_S \mathbf{v} \cdot \mathrm{d}\mathbf{S}. \tag{268}$$

This makes intuitive sense since its basically:

$$\frac{\Delta x}{\Delta t} \cdot \Delta S = \frac{\Delta V}{\Delta t} = \text{Volume of fluid that flows across the surface element } \Delta S \text{ over an infinitesimal time } \Delta t. \tag{269}$$

Furthermore, we have

Theorem 180. If $-S$ denotes a surface with the opposite orientation, then

$$\int\int_{-S} (\mathbf{F} \cdot \mathbf{n})\mathrm{d}S = -\int\int_S (\mathbf{F} \cdot \mathbf{n})\mathrm{d}S. \tag{270}$$

Recall that given a (regular) parametrization $\mathbf{G}(u, v)$ of S, then a normal vector at a point $p = (u_0, v_0)$ on S is given by

$$\mathbf{N}(p) = \frac{\partial \mathbf{G}}{\partial u}(u_0, v_0) \times \frac{\partial \mathbf{G}}{\partial v}(u_0, v_0). \tag{271}$$

(This makes sense since $\mathbf{G} : D \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$, where D is the connected set of interest in which the variables $(u, v) \in D$, and S refers to the corresponding manifold, in which is a surface in this case.)

Note: Regular parametrization in Rogawski book means that the normal vector $\mathbf{N}$ is non-zero for all u, v. Note that this can be weakened to be that the normal vector $\mathbf{N}$ is non-zero for all u, v except on a set of measure zero.

Additional Note: If we just require that our parametrization is indeed a usual parametrization in the sense as discussed in Discussion Supplement 6, then such a condition for a parametrization to be "regular" must then hold since as a parametrization, we require that $[D\mathbf{G}(u,v)]$ is injective for all $(u,v) \in D^\circ$ (parameter domain), in which we can show (using theorems in Linear Algebra) that this is equivalent to the fact that the normal vector is non-vanishing.

Definition 181. An **oriented parametrization** of a surface S is a regular[a] parametrization $G(u,v)$, with the orientation of S determined by the unit normal vector

$$\mathbf{n}(p) = \frac{\mathbf{N}(p)}{||\mathbf{N}(p)||}. \tag{272}$$

Then, given an oriented parametrization, we say that the **positive orientation of** S is in the direction of the normal vector $\mathbf{N}$. Furthermore, $-\mathbf{N}$ gives the negative orientation of S.

[a] See footnote in the remark above.

Theorem 182. Not every surface is orientable, and not every parametrization is an oriented parametrization.

See Challenge Problem Set 4 for an example!

Last but not least, we introduce a method to compute the vector surface integral as follows:

Theorem 183. Let $\mathbf{G}(u,v) : D \to S \subset \mathbb{R}^3$ be a positively oriented parametrization. Then. we have

$$\int\int_S (\mathbf{F}\cdot\mathbf{n})\mathrm{d}S = \int\int_D \mathbf{F}(\mathbf{G}(u,v))\cdot\mathbf{N}(u,v)\,\mathrm{d}u\mathrm{d}v. \tag{273}$$

Remark: This differs a little from the conditions provided in Rogawski book, in which we have already introduced the relevant definition of parametrization. Thus, we have that given a parametrization G, the way in which the required properties are satisfied can be summarized in the table below:

Requirement	How this is satisfied if G is a parametrization
$\mathbf{G}$ is one-one	$\mathbf{G} : D^\circ \to S$ is injective
$\mathbf{G}$ is regular	$[D\mathbf{G}(u,v)]$ is injective for all $(u,v) \in D^\circ$.
Except possibly for points on ∂D	We usually only have to check for $(u,v) \in D^\circ$ and this excludes ∂D since $D^\circ = D \setminus \partial D$.

Note that the expected properties of integrals on manifolds hold. For instance, if $S = S_1 \cup \cdots \cup S_n$ are disjoint (or intersects on a set of volume 0), we have

$$\int\int_S (\mathbf{F}\cdot\mathbf{n})\mathrm{d}S = \int\int_{S_1} (\mathbf{F}\cdot\mathbf{n})\mathrm{d}S + \cdots + \int\int_{S_n} (\mathbf{F}\cdot\mathbf{n})\mathrm{d}S. \tag{274}$$

We shall cover an example on the computation of flux across curves in $\mathbb{R}^2$ and across surfaces in $\mathbb{R}^3$ below.

Example 184. Compute the flux of the vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \tag{275}$$

along a line segment from $(3,0)$ to $(0,3)$, oriented upwards. You do not have to verify completely that any parametrization you provide is indeed one.

Suggested Solution: We first provide the positively oriented parametrization of the line segment, given by

$$\mathbf{r}(t) = \langle 3-t, t\rangle \tag{276}$$

for $0 \leq t \leq 3$. Indeed, this line segment C is oriented upwards. The corresponding flux integral is given by

$$\int_0^3 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{N}(t)\mathrm{d}t. \tag{277}$$

Thus, we proceed as follows:

- $\mathbf{r}'(t) = \langle -1, 1\rangle$.
- $\mathbf{N}(t) = \langle y'(t), -x'(t)\rangle = \langle 1, 1\rangle$.
- $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{N}(t) = \langle (3-t)^2, t^2\rangle \cdot \langle 1, 1\rangle = t^2 + (3-t)^2$.
- Then, we have

$$\int_0^3 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{N}(t)\mathrm{d}t = \int_0^3 t^2 + (3-t)^2 \mathrm{d}t = 18. \tag{278}$$

Example 185. Consider the following vector field in $\mathbb{R}^3$, given by

$$\mathbf{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 3y^5 \\ y + 10xz \\ z - xy \end{pmatrix}. \tag{279}$$

Evaluate the flux across a unit sphere S^2 in $\mathbb{R}^3$, oriented by outward-pointing normal vectors along the sphere. You do not have to verify completely that any parametrization you provide is indeed one.

Suggested Solution: Consider the following parametrization of the unit sphere $G : [0, \pi) \times [0, 2\pi) \to S^2 \subset \mathbb{R}^3$, given by

$$\mathbf{G}\begin{pmatrix} \phi \\ \theta \end{pmatrix} = \begin{pmatrix} \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\phi) \end{pmatrix}. \tag{280}$$

One can compute the normal vector $\mathbf{N}(\phi, \theta)$, given by

$$\begin{aligned} \mathbf{N}(\phi, \theta) &= \frac{\partial \mathbf{G}}{\partial \phi} \times \frac{\partial \mathbf{G}}{\partial \theta} \\ &= \begin{pmatrix} \cos(\phi)\cos(\theta) \\ \cos(\phi)\sin(\theta) \\ \cos(\phi) \end{pmatrix} \times \begin{pmatrix} -\sin(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \sin^2(\phi)\cos(\theta) \\ \sin^2(\phi)\sin(\theta) \\ \sin(\phi)\cos(\phi) \end{pmatrix} \end{aligned} \tag{281}$$

We then proceed to compute $\mathbf{F} \cdot \mathbf{N}$ as follows:

$$\begin{aligned} &\mathbf{F}(\mathbf{G}(\phi, \theta)) \cdot \mathbf{N}(\phi, \theta) \\ &= \begin{pmatrix} \sin(\phi)\cos(\theta) + 3\sin^5(\phi)\sin^5(\theta) \\ \sin(\phi)\sin(\theta) + 10\sin(\phi)\cos(\theta)\cos(\phi) \\ \cos(\phi) - \sin(\phi)\cos(\theta)\sin(\phi)\sin(\theta) \end{pmatrix} \cdot \begin{pmatrix} \sin^2(\phi)\cos(\theta) \\ \sin^2(\phi)\sin(\theta) \\ \sin(\phi)\cos(\phi) \end{pmatrix} \\ &= \sin^3(\phi)\cos^2(\theta) + 3\sin^7(\phi)\sin^5(\theta)\cos(\theta) + \sin^3(\phi)\sin^2(\theta) + 10\sin^3(\phi)\cos(\phi)\cos(\theta)\sin(\theta) \\ &+ \cos^2(\phi)\sin(\phi) - \sin^3(\phi)\cos(\phi)\sin(\theta)\cos(\theta) \\ &= \sin(\phi) + 3\sin^7(\phi)\sin^5(\theta)\cos(\theta) + 10\sin^3(\phi)\cos(\phi)\cos(\theta)\sin(\theta) \\ &- \sin^3(\phi)\cos(\phi)\sin(\theta)\cos(\theta). \end{aligned} \tag{282}$$

Now, note that upon integrating this expression for $\int_0^{2\pi}\int_0^{\pi} \mathbf{F} \cdot \mathbf{N} d\phi d\theta$, note that

- $\int_0^{2\pi} \sin^5(\theta)\cos(\theta) d\theta = \frac{\sin^6(\theta)}{6}\Big|_{\theta=0}^{\theta=2\pi} = 0$, and
- $\int_0^{2\pi} \sin(\theta)\cos(\theta) d\theta = \frac{\sin^2(\theta)}{2}\Big|_{\theta=0}^{\theta=2\pi} = 0$, while
- $\int_0^{2\pi}\int_0^{\pi} \sin(\phi) d\phi d\theta = 2 \times 2\pi = 4\pi$.

Thus, we have

$$\int\int_S (\mathbf{F} \cdot \mathbf{n}) dS = 4\pi. \tag{283}$$

Note that one should check that the parametrization indeed gives the positive orientation of the sphere. This can be done by looking at a specific point, say $\theta = 0$ and $\phi = \pi/2$. Note that this corresponds to the point $(1, 0, 0)$, and the corresponding normal vector is given by $\mathbf{N}(\pi/2, 0) = (1, 0, 0)$, which is precisely parallel to the position vector of the point $(1, 0, 0)$. Thus, we can see that the parametrization is indeed a positively oriented one. If this is not, we can instead use $-\mathbf{N}$ in the corresponding formula for flux (since I can just use the same normal vector $\mathbf{N}$ to be integrated over the surface $-S$ to get the right orientation, and this is just the negative of the flux integral over the original surface S).

Remark 186. Note that the tedious computation can be avoided if we use a tool that is (possibly) going to be covered in the upcoming lecture, mainly, the Divergence Theorem. One can note that $\nabla \cdot \mathbf{F} = 3$, and that the volume of the unit sphere is given by $\frac{4}{3}\pi$, in which $3 \times \frac{4}{3}\pi = 4\pi$. This is actually not a co-incidence! This will be further explained in the next Discussion Supplement.

Summarizing what we have from Discussion Supplement 8 and 9, we have the following types of integrals for $\mathbb{R}^3$:

1. **Scalar line integral** along a curve C given by $\mathbf{r}(t)$ for $a \leq t \leq b$:

$$\int_C f(x, y, z) \mathrm{d}s = \int_a^b f(\mathbf{r}(t)) ||\mathbf{r}'(t)|| \mathrm{d}s. \tag{284}$$

2. **Vector line integral** (i.e work done) along a curve C given by $\mathbf{r}(t)$ for $a \leq t \leq b$:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \mathrm{d}t. \tag{285}$$

3. **Vector line integral** (i.e flux) across a curve C given by $\mathbf{r}(t)$ for $a \leq t \leq b$:

$$\int_C \mathbf{F} \cdot \mathbf{n} \mathrm{d}s = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{N}(t) \mathrm{d}t. \tag{286}$$

4. **Surface integral** over a surface with parametrization $\mathbf{G}(u, v)$ and a parameter domain D:

$$\int \int_S f(x, y, z) \mathrm{d}S = \int \int_D f(\mathbf{G}(u, v)) ||\mathbf{N}(u, v)|| \mathrm{d}u \mathrm{d}v. \tag{287}$$

5. **Vector surface integral** across a surface with parametrization $\mathbf{G}(u, v)$ and a parameter domain D:

$$\int \int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int \int_S \mathbf{F} \cdot \mathbf{n} \mathrm{d}S = \int \int_D \mathbf{F}(\mathbf{G}(u, v)) \cdot \mathbf{N}(u, v) \mathrm{d}u \mathrm{d}v. \tag{288}$$

Exercises:

Exercise 33. Find a negatively oriented parametrization of the parabola $y = -x^2$ for $0 \leq x \leq 1$, oriented from $(1, -1)$ to $(0, 0)$.

Exercise 34. Compute the flux of the vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \tag{289}$$

along the upper half of the unit circle, oriented clockwise.

Exercise 35. Compute the flux of the vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathbf{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ x \end{pmatrix} \tag{290}$$

across the plane $3x - 4y + z = 1$, with $0 < x < 1$ and $0 < y < 1$, oriented with upward-pointing normal.

Exercise 36. Consider the following vector field in $\mathbb{R}^3$, given by

$$\mathbf{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 3y^5 \\ y + 10xz \\ z - xy \end{pmatrix}. \tag{291}$$

Evaluate the flux across an open unit disk with $z = 0$ in $\mathbb{R}^3$ (given by $\{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$), oriented by downward-pointing normal vectors along the disk. You do not have to verify completely that any parametrization you provide is indeed one.

<u>Partial Solutions/Hints:</u>

- Exercise 33. Set $\mathbf{r}(t) = (t, -t^2)$ for $0 \leq t \leq 1$. Note that this goes from $(0, 0)$ to $(1, -1)$, and thus is negatively oriented.
- Exercise 34. Use polar coordinates. Let $\mathbf{r}(\theta) = \langle \cos(\theta), \sin(\theta) \rangle$ for $0 \leq \theta \leq \pi$. You should arrive at the fact that $\mathbf{F} \cdot \mathbf{N} = 0$, and thus the corresponding flux should be 0.
- Exercise 35. The corresponding parametrization is given by $\mathbf{G}(x, y) = \begin{pmatrix} x \\ y \\ 1 - 3x + 4y \end{pmatrix}$. The corresponding normal vector is given by $\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$ (which shouldn't be surprising since given a plane $ax + by + cz = d$, the normal is along the direction $\langle a, b, c \rangle$). This is "upwards" directing since the $z-$component is positive. The flux integral reduces to

$$\int_0^1 \int_0^1 (13x - 13y - 4) \mathrm{d}x \mathrm{d}y = -4. \tag{292}$$

- Exercise 36. Work through this using a polar coordinates (this should be somewhat similar to Example 185). You should obtain that the flux is 0.

10 Discussion 10

Summary for Lectures 22 - 24.

Instead of going upwards from Lecture 22 to 24 in a sequential order, I will first deal with the technicalities of a manifold with boundary, introduce the Stokes' theorem in 3D, and then specializing them to the (Divergence and) Stokes' theorem in 2D (which are both known as Green's theorem).

Definition 187. A subset $M \subset \mathbb{R}^n$ is a **differentiable** $k-$**dimensional manifold with boundary** embedded in $\mathbb{R}^n$ if for every $\mathbf{z} \in M$, we have either:

(i) There exists an open neighborhood $U \subset \mathbb{R}^n$ such that there exists a C^1-mapping $F : U \to \mathbb{R}^{n-k}$ such that

- $M \cap U = \{\mathbf{z} \in U : F(\mathbf{z}) = \mathbf{0}\}$ and
- $[DF(\mathbf{z})]$ is surjective.

OR

(ii) There exists an open neighborhood $V \subset \mathbb{R}^n$ such that there exists a C^1-mapping $F : U \to \mathbb{R}^m$ (with $m \geq n-k$) with the following properties:

- $G(\mathbf{z}) = \mathbf{0}$
- $M \cap V = \{\mathbf{z} \in V : G(\mathbf{z}) \geq \mathbf{0}\}$ and[a]
- $[DG(\mathbf{z})]$ is surjective.

Here, we say that the set of points $\mathbf{z} \in M$ satisfying the second condition to be the boundary of M (denoted also by ∂M, just like how we have discussed about boundary of sets in general).

[a] Inequality here refers to component-wise inequality; ie $\langle x, y\rangle \geq 0$ means $x \geq 0$ and $y \geq 0$.

Example: The upper half-space $H^k \subset \mathbb{R}^k$ is a closed set

$$H^k := \{\mathbf{x} = \langle x_1, ..., x_k\rangle \in \mathbb{R}^k : x_k \geq 0\}. \tag{293}$$

This is a $k-$dimensional manifold with boundary

$$\partial H^k = \{\mathbf{x} = \langle x_1, ..., x_k\rangle \in \mathbb{R}^k : x_k = 0\}. \tag{294}$$

One should take a look at the lecture example on computing the boundary of a "closed" unit cube in $\mathbb{R}^3$ and verifying the appropriate definition.
We continue with the above definition as follows:

Definition 188.

- If $\mathbf{z} \in \partial M$ (ie satisfies the second condition), then we say that $\mathbf{z}$ is a corner point of co-dimension m.
- In the special case of $m = 1$, we then say that $\mathbf{z}$ is in the **smooth** boundary of M ($\partial^s M$).
- The set of corner points that are not in $\partial^s M$ is called the **non-smooth boundary** of M.

We have the following properties for these smooth boundaries as recorded below:

Proposition 189.

- The smooth boundary $\partial^s M$ is a $(k-1)-$dimensional manifold.
- The non-smooth boundary of M has a $(k-1)$-dimensional volume of 0.

Before we present Stokes' theorem, here is a technicality that we have to deal with

Definition 190. For an oriented surface S, the boundary orientation ∂S is chosen such that if your feet is at a point $p \in S$, and your head is where the head of $\mathbf{n}(p)$ is (recall that $\mathbf{n}$ is already determined, given an orientation of the surface S), then the orientation of ∂S is chosen such that S is always to your left.

We then have the Stokes' theorem below:

Theorem 191. (Stokes' Theorem.) Let $G(u, v) : D \to \mathbb{R}^3$ be a positively oriented parametrization of a surface S. The orientation of ∂S is then determined by Definition 190 above. Suppose that $\mathbf{F}$ is a smooth vector field in a solid region W containing S, then

$$\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int\int_S \mathrm{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S}. \tag{295}$$

Definition 192. A **closed** surface is a surface that has no boundary. That is, $\partial S = \emptyset$.

Corollary 193. Let S be a **closed surface**. Then,

$$\int\int_S \mathrm{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S = \emptyset} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0 \tag{296}$$

Example: A unit sphere in $\mathbb{R}^3$ is a closed surface.

Definition 194. Let $\mathbf{F}$ be a vector field defined on a region $W \subset \mathbb{R}^3$. Suppose that $\mathbf{F} = \mathrm{curl}(\mathbf{A})$ for some vector field $\mathbf{A}$ in $\mathbb{R}^3$. Then, we say that $\mathbf{A}$ is the **vector potential** of $\mathbf{F}$ on W.

As a corollary, we have the following computations:

Corollary 195. If $\mathbf{A}$ is a **vector potential** of $\mathbf{F}$ on W, then under the conditions in Theorem 191, we have

$$\int\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int\int_S \mathrm{curl}(\mathbf{A}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S} \mathbf{A} \cdot \mathrm{d}\mathbf{r} = \int\int_{S'} \mathbf{F} \cdot \mathrm{d}\mathbf{S} \tag{297}$$

In other words, the surface integral of $\mathbf{F}$ is surface-independent (ie the same over S and S', provided that they share the same boundary $\partial S = \partial S'$).

Furthermore, we have a corollary of the above corollary below:

Corollary 196. If $\mathbf{F}$ has a vector potential $\mathbf{A}$ on W, and S is a closed surface in W, we then have

$$\int\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S = \emptyset} \mathbf{A} \cdot \mathrm{d}\mathbf{r} = 0. \tag{298}$$

Remark 197.

- For Theorem 191, we are given $\mathbf{F}$ and asked to compute $\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$. Thus, we just compute it using Theorem 191 and the corresponding $\mathrm{curl}(\mathbf{F})$. No vector potentials are involved.
- For Corollary 195, you are asked to compute $\int\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$. To apply the "reverse" of Stokes' theorem (to convert this to a line integral), we will need to find a corresponding vector potential $\mathbf{A}$ to "uncurl $\mathbf{F}$".

In $\mathbb{R}^2$, we have the following specialization (by looking at $\mathbf{F} = \langle F_x, F_y, 0\rangle$):

Theorem 198. (Green's Theorem.) Let D be a region in $\mathbb{R}^2$ such that ∂D is a disjoint union of simple closed curves, with ∂D oriented such that D is always to the left. Suppose that $\mathbf{F} = \langle F_1, F_2\rangle$ and let $\mathbf{F}_z = \langle F_1, F_2, 0\rangle$ (embedding this vector into $\mathbb{R}^3$ with $z = 0$; thus D_z refers to the corresponding $2-$dimensional manifold (with boundary) embedded in $\mathbb{R}^2$), then we have

$$\oint_{\partial D} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int\int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}A = \int\int_{D_z} \mathrm{curl}(\mathbf{F}_z) \cdot \mathrm{d}\mathbf{S}. \tag{299}$$

Since the general form of divergence theorem is not covered till Lecture 25, we shall directly state the equivalent version in $\mathbb{R}^2$ below:

Theorem 199. (Divergence Theorem in $\mathbb{R}^2$ a.k.a Green's Theorem - Flux Form.) Let D be a region in $\mathbb{R}^2$ such that ∂D is a simple closed curve, oriented counterclockwise. Suppose $\mathbf{F} = \langle F_1, F_2\rangle$ is a smooth vector field in D. Then, we have

$$\oint_{\partial D} (\mathbf{F} \cdot \mathbf{n})\, \mathrm{d}s = \int\int_D \mathrm{div}(\mathbf{F})\mathrm{d}A. \tag{300}$$

To end off, we shall include a computational tool related to the Stokes' and Divergence theorems below:

Recall that from Definition 188. we have that the positive orientation of a boundary of a region in $\mathbb{R}^2$ is defined such that as you walk along the curve, the region appears on your left.

Definition 200. A **simple** closed curve C is a closed curve that does not intersect itself.

Theorem 201. (Jordan Curve Theorem.) A simple closed curve C in $\mathbb{R}^2$ splits $\mathbb{R}^2$ into exactly two regions - an interior region D and an exterior region $\mathbb{R}^2 \setminus D$.

Theorem 202. (Addition of Circulation.) Let D be a region in $\mathbb{R}^2$ such that ∂D is a simple closed curve, oriented counterclockwise. If we decompose a domain $D = D_1 \cup D_2$ in which D_1 and D_2 only intersects on their boundaries ∂D_1 and ∂D_2. Then, we have

$$\oint_{\partial D} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_{\partial D_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \oint_{\partial D_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}. \tag{301}$$

With all our concepts in, let us look at a couple of examples!

Example 203. Let $\mathbf{F}$ be a vector field in $\mathbb{R}^2$ given by

$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin(x) \\ x^2y^3 \end{pmatrix}. \tag{302}$$

Use Stokes' Theorem to the circulation of $\mathbf{F}$ over a triangle Δ with vertices $(0,0)$, $(0,2)$ and $(2,2)$.

Suggested Solution: Appeal to Green's Theorem (else you are forced to do three separate line integrals!). We then have

$$\oint_{\partial\Delta} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int\int_{\Delta} \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}\right) \mathrm{d}A. \tag{303}$$

One can compute that

$$\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} = 2xy^3.$$

Then, we have

$$\begin{aligned}\oint_{\partial\Delta} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int\int_{\Delta} \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}\right) \mathrm{d}A \\ &= \int_0^2 \int_{y=0}^{x} 2xy^3 \mathrm{d}y\mathrm{d}x \\ &= \frac{1}{2}\int_0^2 x^5 \mathrm{d}x = \frac{16}{3}.\end{aligned} \tag{304}$$

Here, we note that the triangular region Δ is parametrized by

$$\Delta = \{(x,y)\in\mathbb{R}^2 : 0 \le x \le 2 \text{ and } 0 \le y \le x\}. \tag{305}$$

Example 204. Let $\mathbf{F}$ be a vector field in $\mathbb{R}^3$ given by

$$\mathbf{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3y \\ -2x \\ 3y \end{pmatrix}. \tag{306}$$

Compute the circulation of $\mathbf{F}$ over a circle $x^2 + y^2 = 9$ at $z = 2$ in $\mathbb{R}^3$, oriented **clockwise** as viewed from above.

Suggested Solution: Let D be the circular disk $\{(x, y, 2) \in \mathbb{R}^3 : x^2 + y^2 \leq 9\}$ and C be the aforementioned curve (with the positive orientation as prescribed to be clockwise viewed by someone from above).

Note that Stokes' theorem gives

$$\oint_{\partial D = -C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int\int_D \mathrm{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S}. \tag{307}$$

Note that we have written down $\partial D = -C$. This is because Stokes' Theorem as in Theorem 191 gives an integral along ∂S with orientation chosen by Definition 190, ie such that the surface is always on the left of someone traversing the curve along in the prescribed direction, with that person's head pointing in the positive z direction. This implies that

$$\oint_{\partial C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int\int_D \mathrm{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S}. \tag{308}$$

To compute the surface integral, we first compute the $\mathrm{curl}(\mathbf{F})$, given by

$$\mathrm{curl}(\mathbf{F}) = \begin{pmatrix} 3 \\ 0 \\ -5 \end{pmatrix}. \tag{309}$$

Recall that to evaluate a vector surface integral, we must first parametrize the surface D. Consider the parametrization

$$\mathbf{G}\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \\ 2 \end{pmatrix} \tag{310}$$

with $0 \leq r \leq 3$ and $0 \leq \theta < 2\pi$. One can then compute

$$\frac{\partial \mathbf{G}}{\partial r} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix}, \frac{\partial \mathbf{G}}{\partial \theta} = \begin{pmatrix} -r\sin(\theta) \\ r\cos(\theta) \\ 0 \end{pmatrix}, \text{ and } \frac{\partial \mathbf{G}}{\partial r} \times \frac{\partial \mathbf{G}}{\partial \theta} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}. \tag{311}$$

Thus, we have

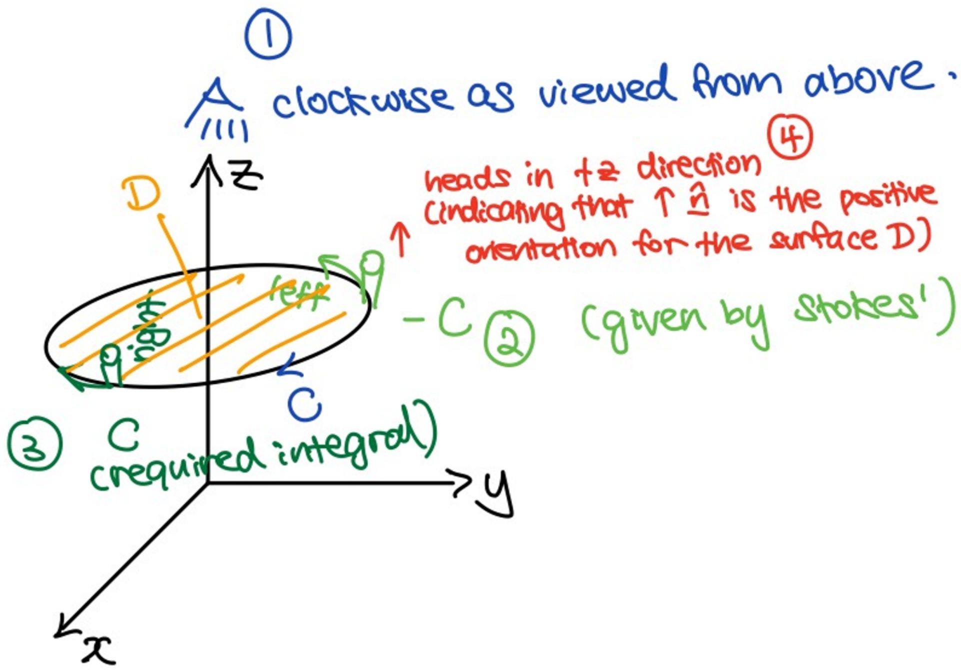
$$\begin{aligned} \int\int_D \mathrm{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} &= \int_0^3 \int_0^{2\pi} \mathrm{curl}(\mathbf{F}) \cdot \left(\frac{\partial \mathbf{G}}{\partial r} \times \frac{\partial \mathbf{G}}{\partial \theta}\right) \mathrm{d}\theta \mathrm{d}r \\ &= \int_0^3 \int_0^{2\pi} -5r \mathrm{d}\theta \mathrm{d}r \\ &= -45\pi. \end{aligned} \tag{312}$$

Note: It is not clear if we should pick $\left(\frac{\partial \mathbf{G}}{\partial r} \times \frac{\partial \mathbf{G}}{\partial \theta}\right)$ (ie the orientation of the parametrization is positive) or $-\left(\frac{\partial \mathbf{G}}{\partial r} \times \frac{\partial \mathbf{G}}{\partial \theta}\right)$ (ie the orientation of the parametrization is negative, and thus we have to reverse its orientation (one possibility is to replace θ by $-\theta$)). However, referring back to Definition 190, to a person standing on the curve with their head pointing in the positive $z-$ direction, the head direction corresponds to the direction of positive orientation (ie the normal that gives the positive orientation). This implies that we will accept the normal vector with a positive $z-$ component, and that is given by $\left(\frac{\partial \mathbf{G}}{\partial r} \times \frac{\partial \mathbf{G}}{\partial \theta}\right)$.

Thus, we have

$$\oint_{\partial C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \boxed{45\pi}. \tag{313}$$

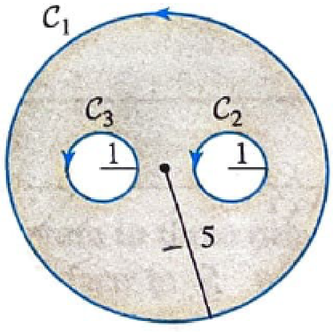
The diagram below helps one to visual this problem in full.



Example 205. (Rogaswki 17.1, Problem 29). Referring to the diagram below, suppose that

$$\oint_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 3\pi, \text{ and } \oint_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 4\pi. \tag{314}$$

Use Green's Theorem to determine the circulation of $\mathbf{F}$ around C_1, assuming that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 9$ on the shaded region.



Suggested Solutions:

By Green's Theorem, we have

$$\oint_{\partial D} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int\int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A = \int\int_D 9\mathrm{d}A = 9\,(25\pi - 2\pi) = 207\pi. \tag{315}$$

For the region D, the boundary can be computed to give $\partial D = C_1 \cup (-C_2) \cup (-C_3)$. Here, as long as one is consistent, you should always have this decomposition (with the same sign). Consistent here refers to picking an orientation for the "surface" D embedded in $\mathbb{R}^3$, in which if we pick the positive orientation to be outwards, then for a person with their head pointing outwards, then the surface D must be to the left. Thus, for say C_2, we note that if we walk around the prescribed C_2, then the surface D appears on that person's right. Thus, for the surface to appear on that person's left, they should traverse in the direction of $-C_2$. Then, we have

$$\begin{aligned} \oint_{\partial D} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \oint_{\partial D} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ &= 207\pi + 3\pi + 4\pi = 214\pi. \end{aligned} \tag{316}$$

Exercises:

Exercise 37. Compute the flux of the vector field $\mathbf{F}(x, y) = \langle x^3, y^3 + y \rangle$ out of the unit circle in $\mathbb{R}^2$.

Exercise 38. Evaluate

$$\int\int_S \mathrm{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} \tag{317}$$

where $\mathbf{F} = \langle x - y, y - z, x - z \rangle$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ with $z \geq 0$, oriented upwards.

Exercise 39. Let S be an oriented surface in $\mathbb{R}^3$, and let $\mathbf{v} \in \mathbb{R}^3$ be a **constant** vector. Let $\mathbf{F}$ be a vector field in $\mathbb{R}^3$ given by $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$. Prove that

$$\int_{\partial S} (\mathbf{v} \times \mathbf{F}) \cdot \mathrm{d}\mathbf{r} = 2 \int_S \mathbf{v} \cdot \mathrm{d}\mathbf{S}. \tag{318}$$

Here, ∂S is oriented as how a boundary of a surface S should be oriented according to Definition 190.

Partial Solutions/Hints:

- Exercise 37. Appeal to Green's Theorem (flux form)/Divergence Theorem. Compute $\operatorname{div}\mathbf{F} = 3x^2+3y^2+1$. The flux is given by $\int\int_{\text{Unit circle}} \operatorname{div}(\mathbf{F})\mathrm{d}A = \int\int_{\text{Unit circle}}(3x^2+3y^2+1)\mathrm{d}A$. Then, change to polar coordinates to obtain $\frac{5\pi}{2}$ as the solution.
- Exercise 38. The boundary of S is the circle C in the $xy-$plane, centered at $(0,0)$ with radius 2 and oriented counterclockwise. Thus, C can be parametrized by $\mathbf{r}(t) = \langle 2\cos(\theta), 2\sin(\theta), 0\rangle$ for $0 \leq \theta \leq 2\pi$. By Stokes' theorem, we have $\int\int_S \operatorname{curl}(\mathbf{F})\cdot \mathrm{d}\mathbf{S} = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_0^{2\pi} 4\sin^2(\theta)\mathrm{d}\theta = 4\pi$.
- Exercise 39. By Stokes' Theorem, this is just a computation question: ie show that $\operatorname{curl}(\mathbf{v}\times\mathbf{F}) = 2\mathbf{v}$. Do this with the given $\mathbf{F}$ and set $\mathbf{v} = \langle a, b, c\rangle$ where $a, b,$ and c does not depend on either $x, y,$ or z.

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